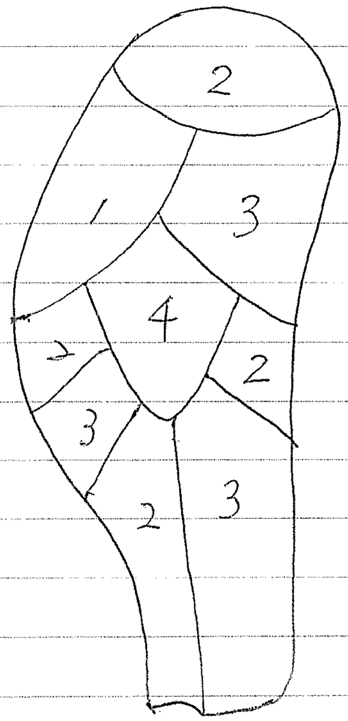
FGT Lecture 6

Nov. 9 —

NO. 6-1
DATE

Graph Coloring



Map Coloring Problem

1852, Oct. 23, Augustus De Morgan $\xrightarrow{\text{sent a letter}}$ William Rowan Hamilto
(University College, London) (Trinity College, Dublin)

. . . My pupil says he guessed $\boxed{\text{4 colors}}$ are enough to color the compartments (of England) differently so that figures with any portion of common boundary are differently colored. . . .

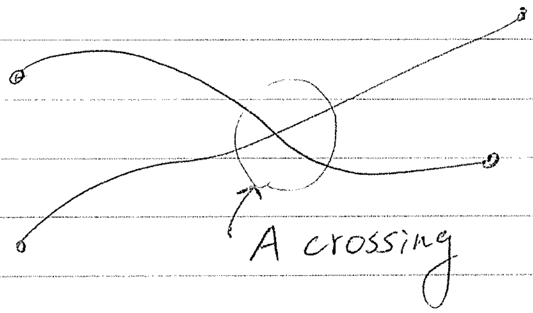
Student : Frederick Guthrie

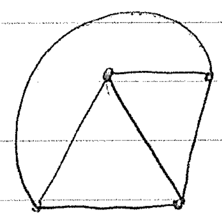
Learned this idea from his brother : Francis Guthrie
(He provided an incomplete proof.)

Definition (Region Colorable)

A plane graph G is said to be n-region colorable if the regions of G can be colored with n or fewer colors so that adjacent regions are colored differently.

Note A plane graph G is a graph which can be drawn on a plane such that no two edges have a crossing.





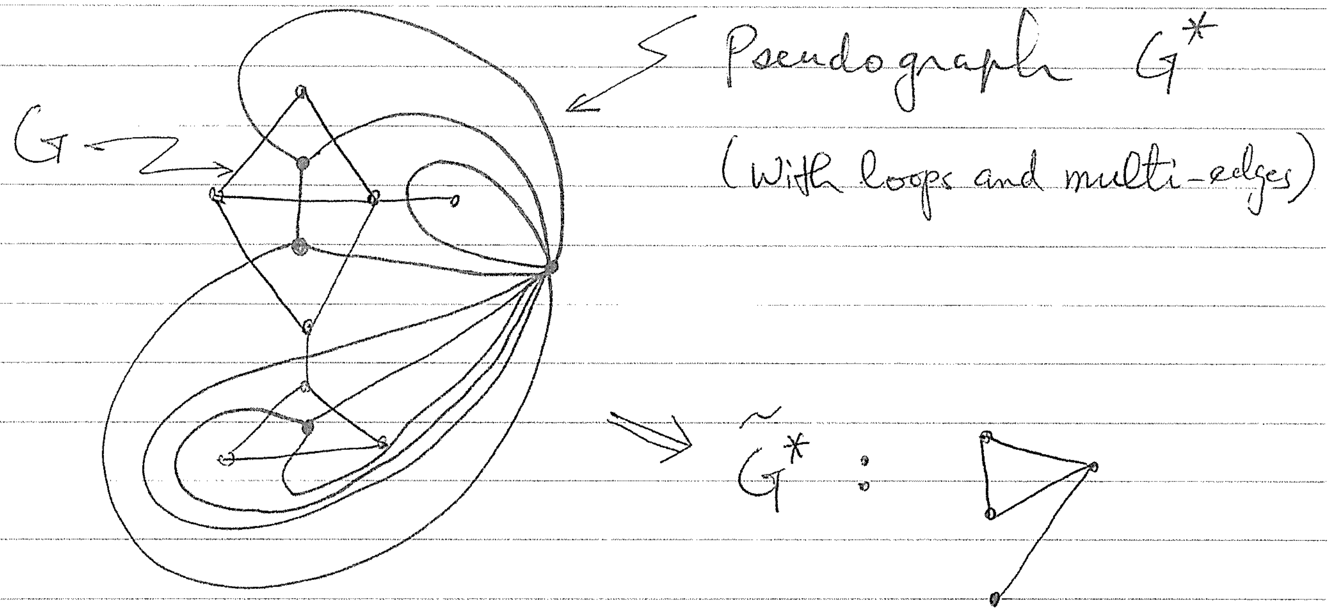
K_4 is a plane graph, but not K_5.

Remark We shall talk about crossings of graphs later.

The Four Color Conjecture

Every map (plane graph) is 4-region colorable.

Now, it is known as the 4-color Theorem, 4CT in short.



<u>Step 1</u> For each region of a plane graph, define a vertex.

<u>Step 2</u> Join two vertices with an edge (or loop) if they have (one) a common boundary.

<u>Step 3</u> This pseudograph (G^*) is known as the dual of the plane graph (G).

<u>Fact</u> $(G^*)^* \cong G$.

<u>Fact</u> $\tilde{G}^*$ is called the <u>underlying graph</u> of G^* if loops are deleted and replace the multiple edges by a single edge.

So, we can use the vertex coloring of $\tilde{G}^*$ to color the regions of a plane graph G.

Definition (Vertex coloring)

A vertex k-coloring φ (k-coloring in short) is a mapping $\varphi: V(G) \to [1,k]$ such that $\varphi(u) \neq \varphi(v)$ if uv is an edge of G. The minimum number k such that G has a k-coloring is called the $\boxed{\text{chromatic number of } G}$ denoted by $\chi(G)$. (If G has a k-coloring, G is said to k-colorable.)

(Fact 1) If $G \supseteq K_k$, then $\chi(G) \geq k$.

(Fact 2) If G is a bipartite graph, then $\chi(G) = 2$ provided G contains at least one edge.

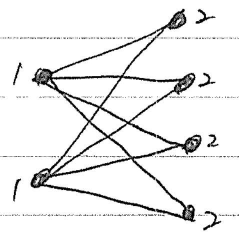
(Fact 3) If G has a k-coloring $\varphi: V(G) \to \{1, 2, \cdots, k\}$, then for each $c \in [1,k]$, $\varphi^{-1}(c) \subseteq V(G)$ is an independent set.

(Fact 4) $\chi(K_n) = n$ and $\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even; and} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$

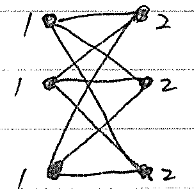
(Fact 5) If $h \geq k$ and G is k-colorable, then G is also h-colorable.

<u>Remark 1</u> If G has a k-coloring φ such that $\left||\varphi^{-1}(c)| - |\varphi^{-1}(d)|\right| \leq 1$ for any $c, d \in [1, k]$, then φ is an equitable k-coloring.

<u>Remark 2</u> If G has a k-coloring, then G "may not" have an equitable k-coloring or equitable $(k+1)$-coloring.



$\chi(K_{2,4}) = 2$



$\chi(K_{3,3}) = 2$

No equitable 3-coloring of $K_{3,3}$ exists. (?)

<u>Definition</u> (critically n-chromatic)

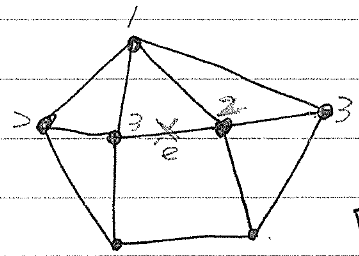
For each $n \geq 2$, G is critically n-chromatic if $\chi(G) = n$ and $\chi(G - v) = n - 1$ for all $v \in V(G)$.

<u>Example</u> K_n is critically n-chromatic.

C_{2k+1} is critically 3-chromatic.

<u>Definition</u> (Minimally n-chromatic)

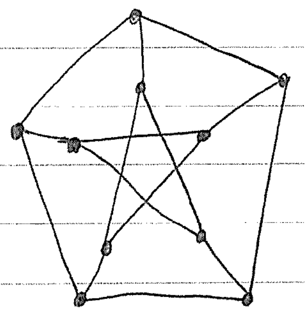
A graph G is minimally n-chromatic if $\chi(G) = n$ and $\chi(G-e) = n-1$ for all edges $e \in E(G)$.



Not minimally 4-chromatic

$\chi(G-e) = 4$

Critically 4-chromatic



Not minimally 3-chromatic

Not critically 3-chromatic

Theorem (Brooks) 1941

If G is a connected graph that is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta(G)$.

Proof. (Omit here.) (*) It is easier to consider the connected graphs which ~~are~~ not regular.

Theorem (Halin 1967, Szekeres and Wilf)

For every graph G, $\chi(G) \leq 1 + \max \delta(G')$, where the maximum is taken over all induced subgraph G' of G. (Notice that if G is not a regular graph, then $\chi(G) \leq 1 + \delta(G) \leq \Delta(G)$.)

Proof. First, we claim that if G is critically n-chromatic, then $\delta(G) \geq n-1$. Suppose not. Let $\delta(G) \leq n-2$ and $\deg(v) = \delta(G)$ where $v \in V(G)$. By assumption, $\chi(G-v) = n-1$. Now, G has an $(n-1)$-coloring since v can be colored with the color missing in its neighbors at most $(n-2)$ vertices. This is a contradition to $\chi(G) = n$.

Now, we are ready for this theorem. Let H be an induced subgraph of G which is n-critical. (We can find this graph by deleting vertices if necessary. So, $\delta(H) \le \max_{G' \le G} \delta(G')$. By the fact that $\delta(H) \ge n-1$, $\max_{G' \le G} \delta(G') \ge n-1 = \chi(G)-1$.

Hence $\chi(G) \le 1 + \max_{G' \le G} \delta(G')$. ∎

<u>Theorem</u> (Gallai, 1968)

For any graph G, $\chi(G) \le 1 + m(G)$, where $m(G)$ denotes the length of a longest path in G. (If G does have a Hamilton path, then $\chi(G)$ is smaller.)

<u>Proof.</u> If H is an n-critical induced subgraph of G, then $\delta(H) \ge n-1$ and therefore H contains a path of length $n-1$. This implies that $m(G) \ge n-1 = \chi(G)-1$ and $\chi(G) \le m(G)+1$. ∎

Greedy Coloring Algorithm

Let $\{1, 2, \cdots, k\}$ be the colors we plan to use. <u>Color a vertex with smaller integer if it is available.</u>