Graph Theory Lecture 8 Nov. 7 –

Theorem 36 $ex(n; K_{s,t}) \le \frac{1}{2}(s-1)^{\frac{1}{t}} \cdot n^{2-\frac{1}{t}} + \frac{1}{2}(t-1)n$, i.e.,

$ex(n; K_{s,t}) \le \frac{1}{2} z(n,n;s,t)$.

Proof. (1st) Let G be an extremal graph such that $\|G\| = ex(n;K_{s,t})$.

Define a bipartite graph $H=(A,B)$ based on G. Let $V(G)=\{v_1, v_2, \cdots, v_n\}$.

Let $A=\{a_1, a_2, \cdots, a_n\}$, $B=\{b_1, b_2, \cdots, b_n\}$ and $a_i \sim_H b_j$ if

and only if $v_i \sim_G v_j$, see Figure 29 for an example. Now,

clearly, $\|H\| = 2\|G\|$ and $a_i \not\sim_H b_i$ for $i=1,2,\cdots,n$. Moreover,

if $G \not\supseteq K_{s,t}$, then $H \not\supseteq K_{s,t}$. (?) This concludes that

$\|G\| = \frac{1}{2}\|H\| \le \frac{1}{2} z(n,n;s,t)$. ■ 1st

(2nd proof) Two-way counting

Consider the number of stars $K_{1,t}$. Since $G \not\supseteq K_{s,t}$,

every set of t vertices of $V(G)$ has at most $s-1$ centers

of stars whose pendant vertices are these vertices. So, there are

at most $(s-1)\binom{n}{t}$ t-stars. The number of stars $K_{1,t}$ can be

obtained by $\sum_{i=1}^{n} \binom{d_i}{t}$ where $d_i = deg_G(v_i)$. Now, let $m = \|G\|$.

$\sum_{i=1}^{n} d_i = 2m$. By a similar technique as Theorem 34, we conclude the proof. ■

$$(*) \quad n \cdot \binom{2m/n}{t} \le (s-1)\binom{n}{t}.$$

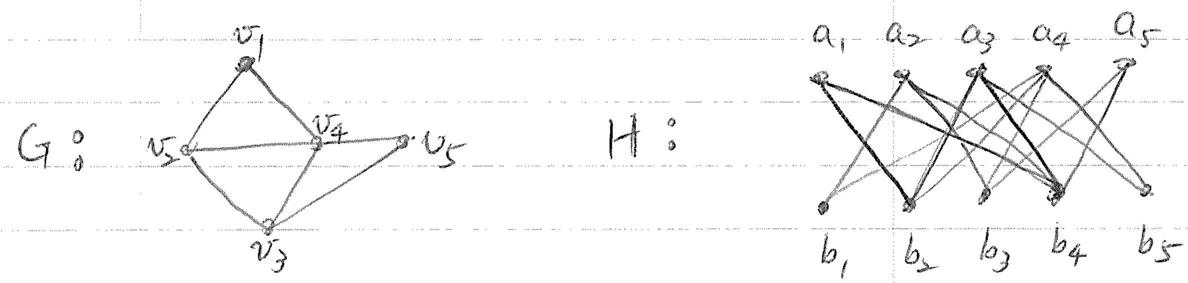


Figure 29. Bipartite version

Ramsey Theory

(◦) The Ramsey number $R(s,t)$ is the smallest value "n" for which either a graph G of order n contains K_s or $\overline{G} \geq K_t$.

(◦) Edge-coloring version of Ramsey number.

The Ramsey number $R(s,t)$ is the smallest value "n" for which in any 2-edge-colored K_n (red and blue), either there exists a red K_s or a blue K_t. (A red K_s is a complete graph of order s such that all its edges are colored red.)

(◦) $R(3,3) = 6$ (Do you know this fact?)

Theorem 37 The following statements are true:

(1) $R(s,2)=s$ and $R(2,t)=t$,

(2) $R(s,t)=R(t,s)$,

(3) For $s>2$, $t>2$, $R(s,t) \le R(s,t-1)+R(s-1,t)$, and

(4) $R(s,t) \le \binom{s+t-2}{s-1} = \binom{s+t-2}{t-1}$.

Proof. (1) and (2) are easy to see.

Claim of (3).

Let $n=R(s,t-1)+R(s-1,t)$. Then, in K_n, each vertex is of degree $R(s,t-1)+R(s-1,t)-1$. Therefore, if K_n is 2-edge-colored by red and blue, then the edges incident to a fixed vertex $x \in V(K_n)$ are either ~~red~~ edges or blue edges. By Pigeon-hole principle, either there are $R(s,t-1)$ blue edges or $R(s-1,t)$ red edges. If the first case holds, then in $\langle N_{K_n}(x)\rangle_{K_n}$ (a complete graph of order $R(s,t-1)$), either there exists a red K_s or blue K_{t-1}. Hence, we have a red K_s or a blue K_t in K_n. The other case can be obtained by a similar argument

<u>Claim of (4)</u> By <u>inductive</u> argument. (Or induction)

$$R(s,t) \le R(s,t-1) + R(s-1,t)$$

$$\le \binom{s+t-1-2}{s-1} + \binom{s-1+t-2}{t-1} = \binom{s+t-3}{s-1} + \binom{s+t-3}{s-2}$$

$$= \binom{s+t-3+1}{s-1} = \binom{s+t-2}{s-1}. \quad \blacksquare$$

<u>Theorem 38</u> (Erdös and Szekeres, 1935)

For each $s \ge 2$, $R(s) \le \dfrac{2^{2s-2}}{s^{1/2}}$. $\quad (R(s) =_{def} R(s,s))$

<u>Proof.</u> $R(s,s) \le \binom{2s-2}{s-1}$. We claim $\binom{2s-2}{s-1} \le \dfrac{2^{2s-2}}{s^{1/2}}$ by induction on s. First, if $s=2$, $2 \le \dfrac{4}{\sqrt{2}}$, the assertion is true. Assume that the assertion is true for $s=k$, thus $\binom{2k-2}{k-1} \le \dfrac{2^{2k-2}}{k^{1/2}}$.

Now, we calculate $\binom{2k}{k} = \dfrac{(2k)!}{k!\,k!} = \dfrac{2k\cdot(2k-1)\cdot(2k-2)!}{k^2\,(k-1)!\cdot(k-1)!}$

$$= \frac{2k(2k-1)}{k^2}\cdot\binom{2k-2}{k-1} \le \frac{4k^2-2k}{k^2}\cdot\frac{2^{2k-2}}{k^{1/2}} = \frac{(4k-2)}{4k}\cdot\frac{2^{2k}}{k^{1/2}}, \quad \text{---} (1)$$

Since $(k+1)^{1/2} \le \dfrac{4k\cdot k^{1/2}}{4k-2}$, we conclude that $\binom{2k}{k} \le \dfrac{2^{2k}}{(k+1)^{1/2}}$. $\blacksquare$

($\circ$) The result has been there for almost 50 years before the improvement due to Thomason in 1988:

$$R(s) \le 2^{2s}/s.$$

(∘∘) The original proof by Ramsey shows that

$$R(s) \leq 2^{2s-3} = \frac{2^{2s-2}}{2}. \quad (1930)$$

Theorem 39 $R(k) \geq \lceil 2^{\frac{k}{2}} \rceil$. $(k \geq 3)$

Proof. (Probabilistic method)

Consider a random red-blue coloring of the edges of K_n. For a fixed set T of k vertices, let A_T be the event that $\langle T \rangle_{K_n}$ is monochromatic. Hence $P(A_T) = \left(\frac{1}{2}\right)^{\binom{k}{2}} \cdot \underset{\substack{\uparrow \\ \text{red or blue}}}{2} = 2^{1-\binom{k}{2}}$.

Since there are $\binom{n}{k}$ possible sets for T, the probability that at least one of A_T occurs is $\binom{n}{k} \cdot 2^{1-\binom{k}{2}}$. Now, if $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$, then no event A_T occurs is of positive probability, i.e., there exists a coloring of edges such that no monochromatic K_k occurs. Therefore, for such n, $R(k) > n$.

Let $n = \lfloor 2^{\frac{k}{2}} \rfloor$. (It suffices to show that $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$.)

$$\binom{n}{k} 2^{1-\binom{k}{2}} \leq \frac{n^k}{k!} \cdot \frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}} \qquad \left(1 - \binom{k}{2} = 1 - \frac{k^2}{2} + \frac{k}{2}\right)$$

$$\leq \frac{\left(2^{\frac{k}{2}}\right)^k}{k!} \cdot \frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}} \leq \frac{2^{1+\frac{k}{2}}}{k!} < 1 \quad (k \geq 3).$$

Hence, $R(k) \geq \lceil 2^{\frac{k}{2}} \rceil$. ▨

(◦) Combining Theorems obtained above

$$2^{\frac{s}{2}} \leq R(s) \leq 2^{2s-3} \quad \text{for } s \geq 2.$$

(◦◦) Open problem: $R(s) = 2^{(c+o(1))s}$, (c may be equal to 1).

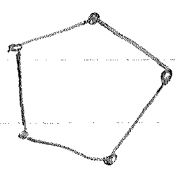
Theorem 40 Known results of $R(s,t)$.

t \ s	3	4	5	6	7	8	9
3	6	9	14	18	23	28	36
4		18	25	36–41	49–61	59–84	73–115
5			43–48	58–87	80–143	101–216	133–316
6				102–165	115–298	134–495	183–780
7					205–540	217–1031	252–1713
8						282–1870	329–3583
9							565–6588

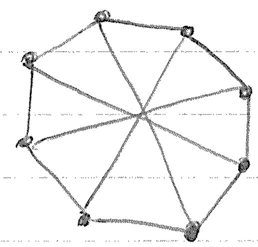
$R(s,t) = R(t,s)$

(◦) The results of lower bounds are obtained by "a special edge-coloring" with two colors. Corresponding to the coloring we have G and $\overline{G}$ of order (prescribed).

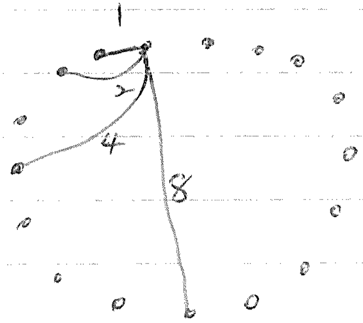
$R(3,3)$: (>5)



$R(3,4)$: (>8)



$R(4,4)$ (>17)



$G(17; \{1,2,4,8\})$

<u>Bonus</u> Find as many vertices (n) as possible such that a graph G of order n satisfying $G \not\supseteq K_5$ and $\overline{G} \not\supseteq K_5$.

(Try 43 !)

<u>Exercise C-1</u> Find a better upper bound for $R(s)$.
(Do your best!)