Graph Theory Lecture 7

10, 31

- A graph F (or a class $\mathcal{F}$) is said to be forbidden in a class of graphs $\mathcal{G}$ if for each $G \in \mathcal{G}$, $G \not\supseteq F$ (or $G \not\supseteq F$ for each $F \in \mathcal{F}$).
- $ex(n; F) = \max\{\|G\| \mid G$ is a graph of order n such that $G \not\supseteq F\}$. $ex(n; \mathcal{F})$ can be defined accordingly.
- The graph G of order n with $\|G\| = ex(n; F)$ is called an extremal graph of order n with forbidden graph F.
- The class of bipartite graphs with partite sets of sizes m and n respectively is denoted by $\mathcal{G}_2(m,n)$.
- The extremal size of graphs in $\mathcal{G}_2(m,n)$ which do not contain $K_{s,t}$ is denoted by $z(m,n; s,t)$. (The notation is in honor of Zarankiewicz.)
- Notice that $ex(n; K_{s,t})$ is different from $z(m,n; s,t)$.
- $z(n,n; s,t) \geq 2\, ex(n; K_{s,t})$. (?)
- $T_r(n) =_{def} K_{\lfloor \frac{n}{r} \rfloor, \lfloor \frac{n}{r} \rfloor, \cdots, \lceil \frac{n}{r} \rceil}$ and $\|T_r(n)\| = t_r(n)$.

Date / / No.

Theorem 31 (Turán, 1941)

$ex(n; K_{r+1}) = t_r(n)$ and $T_r(n)$ is the unique extremal graph.

Proof. (1st) By induction on n. (To show $ex(n; K_{r+1}) = t_r(n)$.)

Since $T_r(n)$ does not contain K_{r+1}, $ex(n; K_{r+1}) \geq t_r(n)$. We claim $ex(n; K_{r+1}) \leq t_r(n)$. Let G be a graph such that $G \not\supseteq K_{r+1}$ and G is of maximum size. Then, $G \supseteq K_r$. For otherwise, we may add more edges to G. Let $W \subseteq V(G)$ and $\langle W \rangle_G \cong K_r$. Let $U = V(G) \setminus W$.

Now, $\|G\| \leq \binom{r}{2} + (r-1)(n-r) + \|\langle U \rangle_G\|$. The term $(r-1)(n-r)$ comes from the fact that each vertex of U is incident to at most $r-1$ vertices of W. By induction hypothesis, $\|\langle U \rangle_G\| \leq t_r(n-r)$. Hence, $\|G\| \leq \binom{r}{2} + (r-1)(n-r) + t_r(n-r) = t_r(n)$. This is a direct consequence of adding one vertex of W to one partite set of $T_r(n-r)$ and $\lfloor \frac{n-r}{r} \rfloor + 1 = \lfloor \frac{n}{r} \rfloor$ $\left(\lceil \frac{n-r}{r} \rceil + 1 = \lceil \frac{n}{r} \rceil \right)$.

Next, we claim the uniqueness. The proof is also by induction on n. Let $y \in V(G)$ such that $\deg_G(y) = \delta(G)$.

Clearly, $G-y$ does not contain K_{r+1} and thus $\|G-y\| = \|G\| - \delta(G) \geq t_r(n-1)$ by the proof of the first part. By induction, $T_r(n-1)$ is the unique graph which is isomorphic to $G-y$. This implies that in $G-y$ the smallest partite set is of size $\lfloor \frac{n-1}{r} \rfloor$. Since $T_r(n-1)$ contains a K_r from r-partite sets, y is incident to at most $r-1$ partite sets of $T_r(n-1)$. Therefore, y can be recognized as a vertex in one of the partite sets, and thus the number of edges between y and $G-y$ is $(n-1) - \lfloor \frac{n-1}{r} \rfloor = n - \lceil \frac{n}{r} \rceil$. This implies that $G \cong T_r(n)$. ∎

(2nd proof) (Zykov) Only $ex(n; K_{r+1}) = t_r(n)$.

Let $v_1 \in V(G)$ such that $\deg(v_1) = \Delta(G)$ and let $W = N(v_1)$. Let $G_1 = G - \langle N(v_1) \rangle_G + T_{r-1}(\Delta(G))$, and $U_1 = V(G_1) \setminus (W \cup \{v_1\})$. See Figure 26.

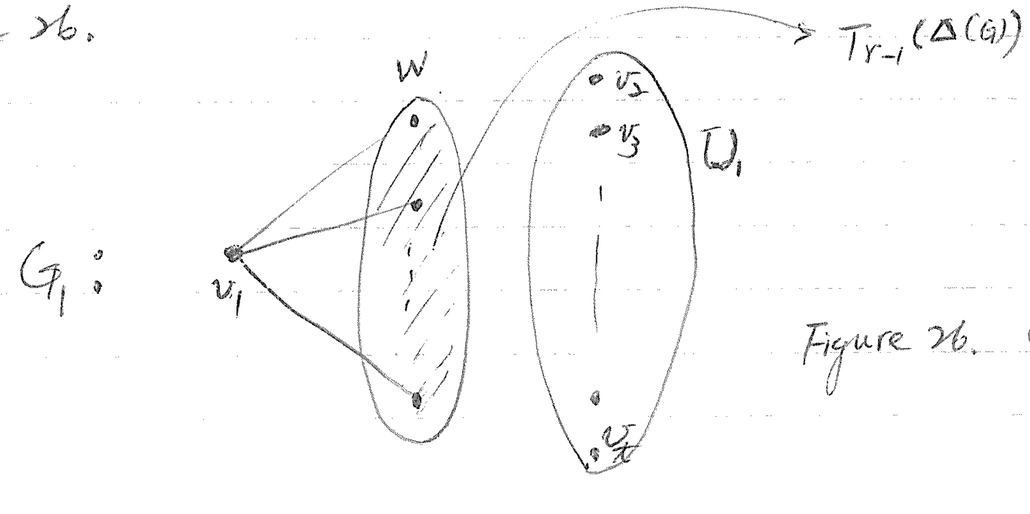

Figure 26. G_1

If U_1 is an empty set, then we stop and evaluate $\|G_1\|$.

Otherwise, if $U_1 \neq \emptyset$, let $v_3 \in U_1$. Now, delete all edges E_2 in G_1 which are incident to v_3 and add v_3u for each $u \in W$ to $G_1 - E_2$ with $U_2 = V(G) \setminus (W \cup \{v_1, v_3\})$.

The new graph is defined as G_2. Since $T_{r-1}(\Delta(G))$ defined on W does not contain K_r, G_2 does not contain K_{r+1}. By continuing this process (until U_t is empty), we shall obtain a complete r-partite graph H such that $\|H\| \geq \cdots \geq \|G_2\| \geq \|G_1\| \geq \|G\|$. (Notice that $\{v_1, v_3, \cdots, v_t\}$ is a new partite set.)

(3rd proof)

We can replace all the vertices of U_1 at the same time by deleting all the edges incident to U_1 and add $\langle W, U_1 \rangle$ to obtain a complete r-partite graph. ∎

Theorem 32 (Erdös, 1970)

Let $G \not\supseteq K_{r+1}$. Then, there exists an H satisfying (1) H is an r-partite graph, (2) $V(H) = V(G)$, and (3) $\forall\ x \in V(G)$, $\deg_G(x) \leq \deg_H(x)$.

Moreover, if G is not a complete r-partite graph, then there exists a vertex $z \in V(G)$, s.t. $\deg_G(z) < \deg_H(z)$.

Proof. By induction on r for the whole statement, and $r=1$ is true. Let the assertion be true for $r' < r$.

Let $x \in V(G)$ s.t. $\deg_G(x) = \Delta(G)$, $N(x) = W$ and $\langle W \rangle_G = G_0$. Clearly, $G_0 \not\supseteq K_r$. By induction, there exists an $(r-1)$-partite graph H_0, s.t., $V(H_0) = W$, $\forall\, y \in W$, $\deg_{G_0}(y) \le \deg_{H_0}(y)$, moreover, if G_0 is not a complete $(r-1)$-partite graph, then there exists a $y' \in W$, s.t. $\deg_{G_0}(y') < \deg_{H_0}(y')$.

Now, let $H = H_0 \underset{(\text{join})}{\vee} (V \setminus W)$, see Figure 27. So, H is an r-partite graph. For $z \in V \setminus W$, $\deg_G(z) \le \Delta(G) = |W| = \deg_H(z)$, and if $z \in W$, $\deg_G(z) \le \deg_{G_0}(z) + n - |W| \le \deg_{H_0}(z) + n - W = \deg_H(z)$. This concludes the first part. For the second part,

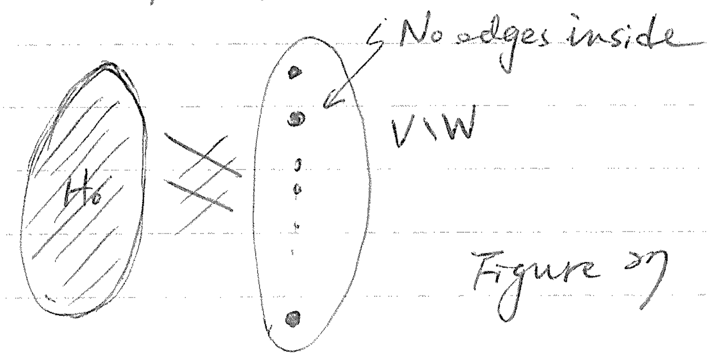

Figure 27

— ——— ——— Assume that $deg_G(x) = deg_H(x)$ for all $x \in V(G)$. Hence, $\|G\| = \|H\|$ and thus $\|G_0\| = \|H_0\|$. (For otherwise, $\|H\| > \|G\|$.) Moreover, $deg_{G_0}(x) = deg_{H_0}(x)$ for each $x \in W$. Suppose not. Let $deg_{G_0}(x') < deg_{H_0}(x')$ for some vertex $x' \in W$. This implies that $deg_G(x') < deg_H(x') = deg_{H_0}(x) + n - |W|$, a contradiction. As a consequence, G_0 (H_0) is a complete $(r-1)$-partite graph and G is a complete r-partite graph as well. ∎

(• •) Try to estimate $z(m,n;s,t)$

<u>Theorem 33</u> (Important Lemma)

Let $2 \le s \le m$, $2 \le t \le n$, $0 \le r \le m$, $z = km + r$ and $z = my$, $y \in \mathbb{R}^+$.
Let G be a bipartite graph, $G \in \mathcal{G}_2(m,n)$ and $G \not\supseteq K_{s,t}$. Then,

$$m \cdot \binom{y}{t} \le (m-r)\binom{k}{t} + r\binom{k+1}{t} \le (s-1)\binom{n}{t}.$$

(Remark. A function $f: \mathbb{R} \to \mathbb{R}$ is convex if $t f(x) + (1-t) f(y) \ge f(xt + y\cdot(1-t))$, $0 \le t \le 1$.)

<u>Proof</u>. Let $G = (A,B)$ where $|A| = m$ and $|B| = n$. Define a graph $H = (A, \binom{B}{t})$. $\binom{B}{t}$ is the collection of all t-subsets of B.

Date / / No.

And $x \sim_H T$ if and only if $x \sim_G y$ for each $y \in T$. Figure 28 is an example $|A| = 5, |B| = 6$ and $t = 3$.

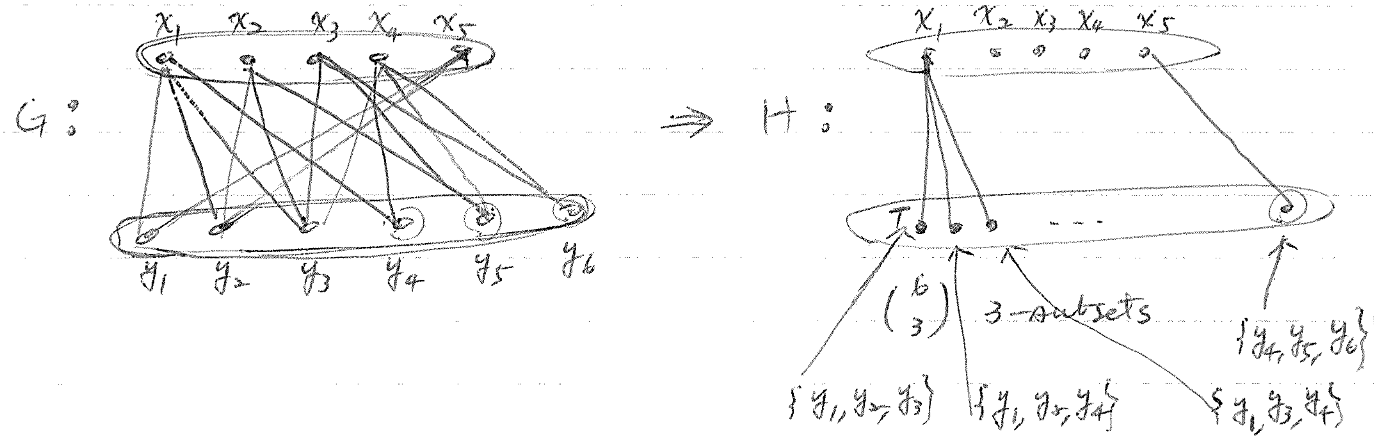

Figure 28 H induced by G

Hence, we have

(1) $\|H\| = \sum_{x \in A} \binom{\deg_G(x)}{t}$, (For example, in Figure 28, $\|H\| = 10$.)

(2) $\forall\ T \in \binom{B}{t}$, $\deg_H(T) \leq s-1$, ($G \not\supseteq K_{s,t}$)

(3) $\|H\| \leq (s-1)\binom{n}{t}$. (From (2).)

Now, since $z = km + r = my = \sum_{x \in A} \deg_G(x)$,

(*) $m\binom{y}{t} \leq (m-r)\binom{k}{t} + r\binom{k+1}{t} \leq \sum_{x \in A} \binom{\deg_G(x)}{t} \leq (s-1)\binom{n}{t}$. ■

(*) comes from the property of combination number. For example, $z = 16, k = 3, m = 5, r = 1$ and $t = 2$. Then, $\overbrace{5 \cdot \binom{3.2}{2}}^{16} \leq \overbrace{4\binom{3}{2} + \binom{4}{2}}^{16} \leq \overbrace{\binom{a}{2} + \binom{b}{2} + \cdots + \binom{c}{2}}^{16}$.

Date / / No.

<u>Theorem 34</u> $z(m,n;s,t) \le (s-1)^{\frac{1}{t}}(n-t+1)\cdot m^{1-\frac{1}{t}} + (t-1)m.$

<u>Proof.</u> By Theorem 33, $m\binom{y}{t} \le (s-1)\binom{n}{t}$, $\dfrac{\binom{y}{t}}{\binom{n}{t}} \le \dfrac{s-1}{m}$.

Hence, $\dfrac{y(y-1)\cdots(y-t+1)}{n(n-1)\cdots(n-t+1)} \le \dfrac{s-1}{m}$.

By the fact $\dfrac{y-i}{n-i} \ge \dfrac{y-t+1}{n-t+1}$ for each $0 \le i \le t-1$,

we have $\left(\dfrac{y-t+1}{n-t+1}\right)^t \le \dfrac{s-1}{m}$, i.e., $(y-t+1)^t \le (s-1)\cdot(n-t+1)^t\cdot m^{-1}$.

This implies that $y-t+1 \le (s-1)^{\frac{1}{t}}\cdot(n-t+1)\cdot m^{-\frac{1}{t}}$ and

$$y \le (s-1)^{\frac{1}{t}}\cdot(n-t+1)\cdot m^{-\frac{1}{t}} + (t-1).$$

Hence, $z = my \le (s-1)^{\frac{1}{t}}(n-t+1)\cdot m^{1-\frac{1}{t}} + (t-1)m.$ ■

<u>Theorem 35-1</u> $z(n,n;2,2) \le \frac{n}{2}\left[1+(4n-3)^{\frac{1}{2}}\right]$.

<u>Proof.</u> By Theorem 33, $n\cdot\binom{y}{2} \le \binom{n}{2}$.

Hence, $n\cdot y(y-1) \le n(n-1)$ and we have $y^2 - y - (n-1) \le 0$.

A direct calculation shows that $y \le \dfrac{1+\sqrt{4n-3}}{2}$. This implies

that $z = m\cdot y = n\cdot y \le \dfrac{n(1+\sqrt{4n-3})}{2}$. ■

<u>Theorem 35-2</u> If $n = q^2+q+1$ and q is a prime power,

then $z(n,n;2,2) = \frac{n}{2}\left[1+(4n-3)^{\frac{1}{2}}\right]$. (Proof. By the existence of a projective plane of order q.)