Lecture 6 Oct. 19 – 06 P2
(Graph Theory)

Theorem 26 (König)

Every r-regular bipartite graph contains r edge-disjoint perfect matchings.

Proof. By induction on r. Clearly, it is true for $r=1$. Let $r \geq 2$. Let G be the r-regular bipartite graph where $G=(A,B)$. Then $|A|=|B|$. So, it suffices to find a matching saturates A. Now, for any subset S of A, $\Gamma(S) = \bigcup_{x \in S} N_G(x)$. If $|S|=k$, then S is incident to $k \cdot r$ edges. Since each vertex of B is of degree r, it takes at least k vertices of B to join with these $r \cdot k$ edges. This implies that $|\Gamma(S)| \geq |S|$. So, by Hall's Theorem, a matching saturates A can be obtained. Following the same process, we conclude the proof. ∎

Theorem 27

Let $G=(A,B)$ be a bipartite graph such that for each $S \subseteq A$, $|\Gamma(S)| \geq |S| - d$, $d < |A|$. Then, G contains a matching with $|A|-d$ edges.

Proof. Clearly, if $d=0$, then we have a matching with $|A|$ edges.

Now, let $d>0$ and $B' = B \cup D$ where $D = \{y_1, y_2, \cdots, y_d\}$ and $D \cap B = \phi$.

Let $G' \doteq (A, B')$ such that $E(G') = E(G) \cup \{y_i a_j \mid i=1,2,\cdots,d;\ j=1,2,\cdots,|A|\}$.

(Join each vertex in D to every vertex of A.)

Now, for each $S \subseteq A$, $|\Gamma(S)| \geq S$ (in G'). Hence G' has a matching saturates A. This implies that G has a matching of size at least $|A|-d$. ∎

<u>Remark</u> The following results can be obtained by applying Hall's Theorem.

1. A Latin rectangle can be extended to a Latin square.
2. An $n \times n$ matrix $A = (a_{ij})$ (non-negative) is said to be doubly stochastic if $\sum_{i=1}^{n} a_{ij} = 1$ for every j and $\sum_{j=1}^{n} a_{ij}$ for every i. Then, there exist $\lambda_k \geq 0$, $\sum_{k=1}^{m} \lambda_k = 1$, and permutation matrices $P_1, P_2, \cdots, P_m$ such that $A = \sum_{k=1}^{m} \lambda_k P_k$.
3. More ...

(0) A 1-factor of a graph G is a 1-regular spanning subgraph of G.

Theorem 28 (Tutte's 1-factor theorem)

A nontrivial graph G has a 1-factor if and only if for every proper subset S of $V(G)$, the number of odd components of $G-S$, $o(G-S) \le |S|$.

Proof. ($\Rightarrow$) Assume that F is a 1-factor of G and there exicts a proper subset W of $V(G)$ such that $o(G-W) > |W|$. Since an odd component H has an odd number of vertices, one of the vertices in H incident to F must be joining a vertex of W. But, we have more odd components than $|W|$. One of the vertices in W will be incident to at least two edges in F, a contradiction.

($\Leftarrow$) Since $o(G-\phi) \le 0$, G contains only even components. Hence, $|G|$ is even. Furthermore, if $|S|$ is odd (even), $o(G-S)$ must be odd (even). So, $|S|$ and $o(G-S)$ are of the same parity.

We shall prove the sufficiency by induction on $|G| = n$.

Clearly, if $n=2$, then $G \cong K_2$. Assume for all graphs H of even order less than n that if $o(H-W) \le |W|$ for every proper subset of $V(H)$, then H has a 1-factor. Let G be a graph of order n and $o(G-S) \le |S|$ for each proper subset S of $V(G)$. We claim that G has a 1-factor.

Case 1. $\forall\ S \subsetneq V(G)$, $|S| \ge 2$ and $o(G-S) < |S|$. —— $(*)$

The fact of parity shows that $o(G-S) \le |S|-2$ for all S. Let $e = uv$ be an edge of G and consider $G-\{u,v\}$. By the fact that: for each proper subset T of $V(G-\{u,v\})$, $o(G-\{u,v\}-T) \le |T|$ and induction hypothesis, $G-\{u,v\}$ has a 1-factor, so is G. (If $o(G-\{u,v\}-T) > |T| = |T \cup \{u,v\}|-2$, then $o(G-(\underbrace{\{u,v\} \cup T}_{S})) \ge |\underbrace{T \cup \{u,v\}}_{S}|$, a contradiction to $(*)$.)

Case 2. $\exists\ R \subseteq V(G)$, s.t. $o(G-R) = |R|$ where $2 \le |R| < n$.

Among all such R's, let S be the one of maximum cardinality $|S| = h$. Now, let $G_1, G_2, \cdots, G_h$ denote the odd components of $G-S$.

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Note that those k odd components are the only components in $G-S$. For otherwise, let G_0 be an even component of $G-S$ and $v_0 \in V(G_0)$. Then, $o(G-S\cup\{v_0\}) \geq h+1 = |S\cup\{v_0\}|$. In fact, $o(G-S\cup\{v_0\}) = |S\cup\{v_0\}|$ by the assumption. Now, we have a larger "R" for S, a contradiction.

For $i=1,2,\cdots,h$, let S_i be the set of vertices in S which are adjacent with vertices in G_i. None of S_i's will be empty. For otherwise, G_i is an odd component of G and it is not possible. (G has only even components.)

Now, for $1 \leq k \leq h$, consider the union T of "any" k sets in $\{S_1, S_2, \cdots, S_h\}$. Suppose that $|T| < k$. Since $o(G-T)$ is at least k, $o(G-T) \geq k > |T|$ which violates the assumption $o(G-S) \leq |S|$. So, $\{S_1, S_2, \cdots, S_h\}$ has an SDR $(v_1, v_2, \cdots, v_h)$ where $v_i \in S_i$. Moreover, in G_i, let $u_i \sim_G v_i$.

For showing that G has a 1-factor, it's left to show that for each $i = 1, 2, \cdots, h$, $G_i - u_i$ has a 1-factor.

Therefore, let W be a proper subset of $V(G_i - u_i)$ and we claim $o(G - u_i - W) \le |W|$. (This will imply the existence by induction.)

Suppose not. Let $o(G - u_i - W) > |W|$. Again, since $o(G - u_i - W)$ and $|W|$ are of the same parity, we have $o(G - u_i - W) \ge |W| + 2$. Now, combining with S,

$$o(G - u_i - W - S) = o(G - S) + o(G - u_i - W) - 1$$

$$\ge |S| + |W| + 2 - 1$$

$$= |S| + |W| + 1 = |\{u_0\} \cup W \cup S|.$$

Hence, we conclude that $o(G - (\{u_0\} \cup W \cup S)) = |\{u_0\} \cup W \cup S|$.

Since $\{u_0\} \cup W \cup S$ is larger than S (in size), this contradicts to the choice of S. As a consequence, we have the fact: $G_i - u_i$ contains a 1-factor and thus G has a 1-factor. ∎

Theorem 29 (Petersen) Every 2-edge-connected (bridgeless) cubic graph G has a 1-factor F and $G - F$ is a 2-factor.

Proof. Let $S \subseteq V(G)$ and consider an odd component in $G-S$. (Notice that if $o(G-S) = 0$, then $0 \leq |S|$.) Since G is cubic, the number of edges between S and C must be odd. (Otherwise, the degree sum of $V(C)$ in $G-S$ is odd.) By the assumption that G is 2-edge-connected, there are at least three edges in $\langle S, C \rangle$. This implies that the total edges between S and $G-S$ is at least $3 \cdot o(G-S)$. By the fact that G is cubic, such edges are at most $3 \cdot |S|$. Hence, $3 \cdot o(G-S) \leq 3 \cdot |S|$. By Tutte's 1-factor theorem, G has a 1-factor F and $G-F$ is clearly a 2-factor.

$\blacksquare$

Theorem 30 (Petersen's 2-factor theorem)

Let k be an even integer. Then, a k-regular graph contains $\frac{k}{2}$ edge-disjoint 2-factors.

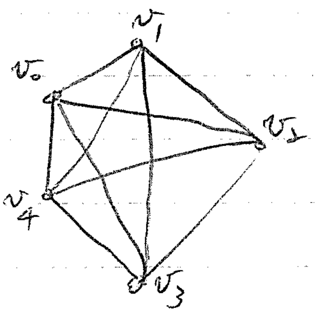
Proof. It suffices to consider a connected k-regular graph G. Let $k = 2h$. By Euler's circuit theorem, G has an eulerian circuit Z : $\langle\langle v_0, v_1, v_2, \cdots, v_i, v_{i+1}, \cdots, v_t, v_0 \rangle\rangle$.

Now, we defined a bipartite graph $\tilde{G} = (A,B)$, such that $|A| = |B| = |G|$, (See Figure 26.) and $v_i \underset{\tilde{G}}{\sim} u_j$ if v_i, v_j are two consecutive vertices in Z. Since G is $2h$-regular, $\tilde{G}$ is h-regular. By König's Theorem, G contains h edge-disjoint perfect matchings. It is not difficult to see that a perfect matching in $\tilde{G}$ gives a 2-factor in G. This concludes the proof. ∎

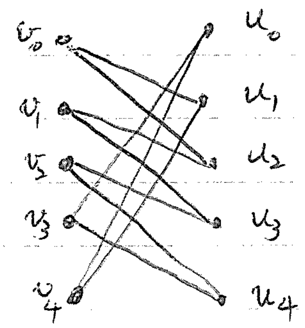
(∘) Unfortunately, we are not able to control the type of 2-factors we are going to obtain.

(∘∘) A perfect matching in $\tilde{G}$ can be represented as a permutation.

G :



$\tilde{G}$:



$((v_0, v_1, v_2, v_3, v_4, v_0, v_2, v_4, v_1, v_3))$

is an eulerian circuit of G.