Graph Theory Lecture 5 (Thm. 21–25)

Oct. 12 –

Review

Definition (Network)

A network N is a digraph D with two distinguished vertices u and v, called the source and sink of N, respectively, and a non-negative integer-valued function c on $E(D)$. The digraph is the underlying digraph of N and the function c is the capacity function on N. For convenience, $c(\vec{a}) = c((x,y)) = c(x,y)$ for each arc $\vec{a} = (x,y)$, is the capacity of $\vec{a}$.

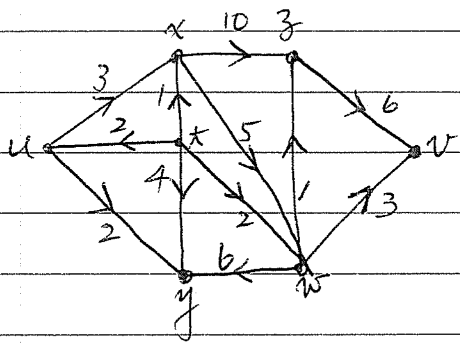


Figure 20. A network.

$$N^+(x) = \{y \mid y \in V(D) \text{ and } (x,y) \in E(D)\}$$

$$N^-(x) = \{y \mid y \in V(D) \text{ and } (y,x) \in E(D)\}$$

(•) A flow in a network N, f, is a function on $E(D)$, s.t.

① $0 \le f(\vec{a}) \le c(\vec{a})$ for every $\vec{a} \in E(D)$, and
(capacity bound)

② $\sum_{y \in N^+(x)} f(x,y) = \sum_{y \in N^-(x)} f(y,x)$ for every $x \in V(D) \setminus \{u,v\}$.
(Conservation law)

- The net flow into x is equal to $\sum_{y \in N^-(x)} f(y,x) - \sum_{y \in N^+(x)} f(x,y)$ which is zero except $x \in \{u,v\}$.

- $(V_1, V_2) = \{(x,y) \in E(D) \mid x \in V_1 \text{ and } y \in V_2\}$ (digraph version!)

(∘∘) A cut in N is $(X, \underbrace{V(D) \setminus X}_{= X'})$ such that $u \in X$ and $v \in V(D) \setminus X$.

<u>Definition</u> Let $K = (X, X')$ be a cut in N. Then, the capacity of K, $\operatorname{cap} K = c(X, X') = \sum_{(x,y) \in K} c(x,y)$.

e.g. In Figure 20, let $X = \{u, x, t, y\}$, then $c(X, X') = 17$.

(*) <u>Definition</u> The value of a flow f in N is defined as the net value flow out the source and therefore the net value flow into the sink. (Denoted by $\operatorname{val} f$)

<u>Theorem 21</u> Let f be a flow in a network N and $K = (X, X')$ be a cut in N. Then, $\operatorname{val} f \le \operatorname{cap} K$.

Proof. Note that $\operatorname{val} f = f(u, \{u\}') - f(\{u\}', u)$ and

$$f(x, \{x\}') - f(\{x\}', x) = 0, \quad \forall x \in X \setminus \{u\}. \quad (u \in X,\ v \in X')$$

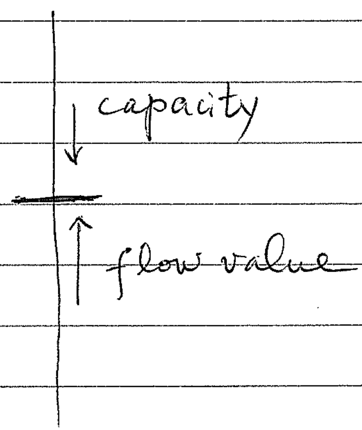
This implies that

$$\underbrace{\sum_{x \in X} \left[f(x, \{x\}') - f(\{x\}', x) \right]}_{= f(X, X') - f(X', X) \ \text{(Calculation)}} = \operatorname{val} f.$$

Now, $f(X, X') \le \operatorname{cap}(X, X')$ and $f(X', X) \ge 0$. Hence,

$\operatorname{val} f \le \operatorname{cap} K$.

- A minimum cut is a cut K in N such that for every cut K' in N, $\operatorname{cap} K \le \operatorname{cap} K'$.
- A maximum flow is a flow f in N such that for every flow f' in N, $\operatorname{val} f \ge \operatorname{val} f'$.



$\Rightarrow$ If there exists a K and an f s.t. $\operatorname{cap} K = \operatorname{val} f$, then K is a minimum cut and f is a maximum flow.

min-max problem

Theorem 22 (Ford and Fulkson, 1956–1962)

In any network N defined on D, the value of a maximum flow equals the the capacity of a minimum cut.

Proof. Clearly, if there exist no cuts such that its capacity of the cut is val f, then f does not exist. (Theorem 21) So, it suffices to claim that if the value of a maximum flow f is v, then there exists a cut K, such that $\operatorname{cap} K = \operatorname{val} f = v$.

Define a subset $S \subseteq V(D)$ recursively as follows. Let $s \in S$. If $x \in S$, and $c(x,y) > f(x,y)$ or $f(y,x) > 0$, then let $y \in S$. We shall prove that (S, S') is a cut with capacity v.

First, we claim $t \notin S$. Suppose not, i.e., $t \in S$. Hence, we can find a sequence of vertices in N such that $s = x_0, x_1, \cdots, x_l = t$. Moreover, if we let $\varepsilon_i = \max\{c(x_i, x_{i+1}) - f(x_i, x_{i+1}), f(x_{i+1}, x_i)\}$, $i = 0, 1, \cdots, l-1$, then $\varepsilon_i > 0$. Let $\varepsilon = \min_i \{\varepsilon_i\}$. Now, let

$f^*(x_i, x_{i+1}) = f(x_i, x_{i+1}) + \varepsilon$ if $c(x_i, x_{i+1}) - f(x_i, x_{i+1}) = \varepsilon_i > 0$ and

$f^*(x_{i+1}, x_i) = f(x_{i+1}, x_i) - \varepsilon$ if $f(x_{i+1}, x_i) = \varepsilon_i > 0$. As a consequence, f^* is a flow from s into t with value $\mathrm{val} f^* = v + \varepsilon$, a contradiction. (See Figure 21) By the definition of a flow,

$$\mathrm{val}\, f = v = \sum_{x \in S,\, y \in S'} f(x,y) - \sum_{x \in S',\, y \in S} f(x,y). \quad \text{——— (1)}$$

Again, by the definition of S, if $x \in S$ and $y \in S'$, then $c(x,y) = f(x,y)$ and $f(y,x) = 0$. This implies that $(1) = \underbrace{\sum_{x \in S,\, y \in S'} c(x,y)}_{v \,=\, \mathrm{cap}(S, S')}$, the proof follows. ■

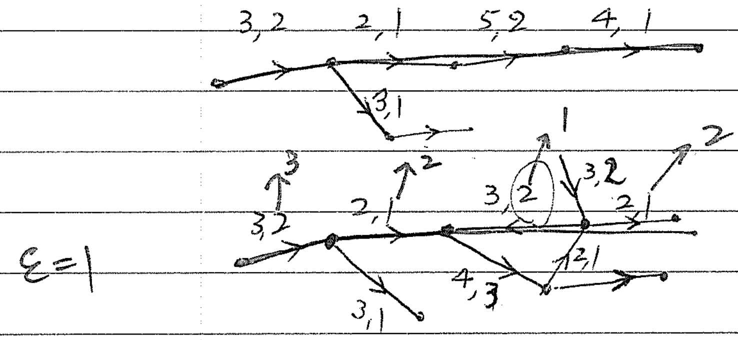


Figure 21 augmenting flow f^*

Theorem 23 (Menger, 1927)

Let s and t be two nonadjacent vertices of a graph G. Then, the minimal number of vertices separating s from t is equal to the maximal number of vertex-disjoint s-t paths. (Internally)

Proof. (1st)

Let the number of vertices separating s and t be k. Then, it is easy to see that there are at most k independent (vertex-disjoint) paths connecting s and t. Also, if $k=1$, then we have a path joining s and t. Now, suppose the assertion is not true, i.e., we can find less than k independent $s-t$ paths for certain k. Now, take the minimal k in which we have a counterexample. Then, among all such examples, let G be the one with minimum size (number of edges).

First, we notice that s and t have no common neighbor. For otherwise, let sx and xt be edges of G. Then, $G-x$ will be a counterexample for "$k-1$" (smaller than k) at most $k-1$ independent paths and.

Let W be a separating set of s and t and $|W|=k$. Suppose, neither $N_G(s)=W$ nor $N_G(t)=W$. (Figure 22-1)

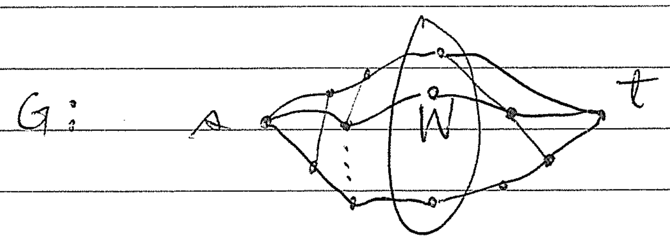


Figure 22-1.

Let G_s be obtained by deleting all the vertices to the left of G in Figure 22-1 and adding a replacing s' with edges joining to W, see Figure 22-2. Now, G_s has fewer edges than G and thus there are k independent s'-t paths. Hence, we have k W-t independent paths. With the same technique, we derive k s-W independent paths (by changing s to t).

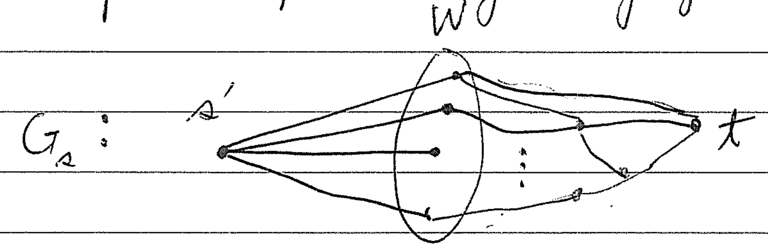


Figure 22-2.

So, as a conclusion, either s or t must have their neighbors W. Let $N_G(s) = W$ and $P = \langle s, x_1, x_2, \cdots, x_\ell, t \rangle$ be a shortest s-t path. Then, $\ell \geq 2$. Consider $G - x_1 x_2$.

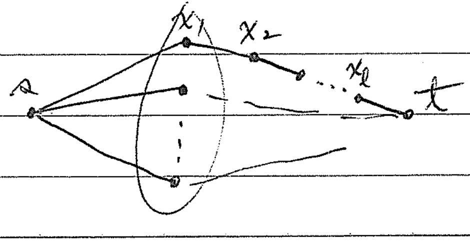


Figure 22-3.

In $G - x_1 x_2$, there exists ~~an~~ s-t separating set W_0 of size $k-1$. Then,
both $W_1 = W_0 \cup \{x_1\}$ and $W_2 = W_0 \cup \{x_2\}$ are s-t separating sets of G.

By the fact that P is a shortest s-t path, s is not adjacent to x_2 and t is not adjacent to x_1. This implies that $N_G(s) = W_1$ since t is not adjacent to a vertex of the separating set W_1.

Similarly, $N_G(t) = W_2$. Hence, $\underline{N_G(s) \cap N_G(t) = W_0}$, a contradiction.
(s and t have common neighbors.)
($|W_0| = k-1 \geq 1$.) ▨

(2nd proof.) Exercise B-1.

(∘) The "Edge" version of Menger's theorem can be stated as follows:

Let s and t be two vertices of G. Then, the minimal number of edges separating s from t is equal to the maximal number of edge-disjoint s-t paths. (We can prove this part by using Theorem 22. Replace each edge xy of G by (x,y) and (y,x) and assign capacity "1" to each arcs.)

Theorem 24

If S and T are arbitrary subsets of $V(G)$, then the maximal number of vertex-disjoint (including endvertices) $S-T$ paths is $\min\{|W| \mid W \subseteq V(G)$ and $G-W$ has no $S-T$ paths$\}$.

Proof. By adding to new vertices s and t as in Figure 23, we have a new graph $\tilde{G}$. Now, by Menger's Theorem, the maximal number of $S-T$ paths is the same as that of $s-t$ paths in $\tilde{G}$. Hence, we have the proof. ∎

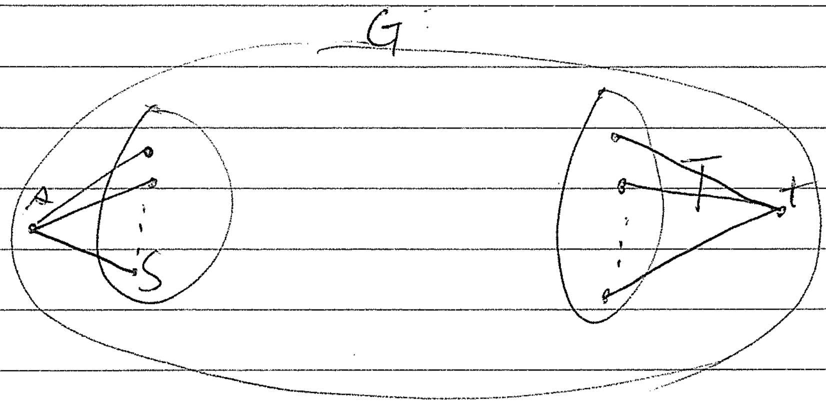


Figure 23, graph $\tilde{G}$

Review

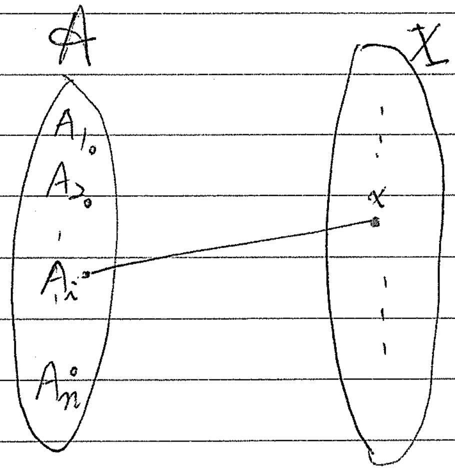
System of Distinct Representatives, SDR

Definition Let $\mathcal{A} = \{A_1, A_2, \cdots, A_n\}$ be a collection of subsets of a given set X. Then, an ordered n-tuple $(a_1, a_2, \cdots, a_n)$ is called an SDR of $\mathcal{A}$ if $a_i \in A_i$, $i = 1, 2, \cdots, n$ and all elements a_i's are distinct.

Hall's Theorem (1935)

$\mathcal{A} = \{A_1, A_2, \cdots, A_n\}$ has an SDR if and only if for each $1 \le k \le n$, the union of any k subsets in $\mathcal{A}$ contains at least k distinct elements, i.e., $\left|\bigcup_{j=1}^{k} A_j\right| \ge k$. (Hall's condition)

We can use a bipartite graph to depict the above idea.



x is incident to A_i iff $x \in A_i$.

Figure 24 (Marriage problem)

<u>Theorem 25.</u> A bipartite graph $G=(A,B)$ contains a matching saturates A if and only if for every $S \subseteq A$, $\Gamma(S) = \bigcup_{x \in S} N_G(x)$ contains at least $|S|$ elements of B, i.e., $|\Gamma(S)| \geq |S|$.

<u>Proof.</u> (1st) ($\Rightarrow$) By the existence of a matching saturates A.

($\Leftarrow$) By Theorem 24, it suffices to prove that there are $|A|$ vertex-disjoint A–B paths (and thus a matching saturates A). Suppose not. Then, there exists a subset A_1 of A and a subset B_1 of B such that there is no edge between $A \setminus A_1$ and $B \setminus B_1$, see Figure 24, and $|A_1| + |B_1| < |A|$. (The number of $A-B$ paths is less than $|A|$.) Hence, there are no edges between $A \setminus A_1$ and $B \setminus B_1$, equivalently $\Gamma(A \setminus A_1) \subseteq B_1$. Then, $|\Gamma(A \setminus A_1)| \leq |B_1| < |A| - |A_1| = |A \setminus A_1|$. $\rightarrow\leftarrow$ ■

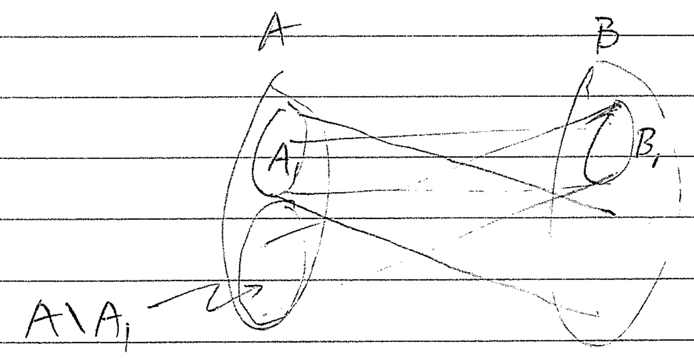


Figure 24.

2nd proof. By induction on $|A|$. Clearly, it's true for $|A|=1$.

First, if for each $S \subsetneq A$, $|\Gamma(S)| \geq |S|+1$, then let $a_1 \in A$ and $a_1b_1 \in E(G)$ where $b_1 \in N_G(a_1)$. Now, consider the bipartite graph $G' = (A_1\setminus\{a_1\}, B_1\setminus\{b_1\})$. Since for each $S' \subseteq A_1\setminus\{a_1\}$, $\Gamma(S') \geq |S'|$, there exists a matching saturates $A_1\setminus\{a_1\}$. Combining with a_1b_1, we have a matching needed.

Second, if there exists a proper subset S of A such that $|\Gamma(S)| = |S|$. By induction, we have a matching M_S saturates S. Now, consider $(A-S, B-T)$ where T is the set of vertices used in M_S. If there exists an $S' \subseteq A-S$ such that $|\Gamma(S')| < |S'|$, then $|\Gamma(S \cup S')| < |S \cup S'|$ a contradiction. Hence, the Hall's condition holds for the graph $(A-S, B-T)$. By induction, we have a matching saturates $A-S$. As a consequence, G has a matching saturates A.

3rd proof. (Rado)

Let $\underset{(A,B)}{\overset{\shortparallel}{G}}$ be a minimal (size) graph satisfying the condition.

It suffices to claim that G contains $|A|$ independent edges.
(matching of size $|A|$.)

Suppose not. There exists two vertices a_1 and a_2 in A and b in B such that a_1b and a_2b are edges of G. Since both $G-a_1b$ and $G-a_2b$ violate Hall's condition, there exist two subsets A_1 and A_2 of A such that $|\Gamma(A_1)|=|A_1|$, $|\Gamma(A_2)|=|A_2|$ and a_i is the only vertex of A_i which is adjacent to b.
$(i=1,2)$

Hence, $$|\Gamma(A_1)\cap\Gamma(A_2)| \geq |\Gamma(A_1-a_1)\cap\Gamma(A_2-a_2)|+1$$
$$\geq |\Gamma(A_1\cap A_2)|+1 \geq |A_1\cap A_2|+1.$$

On the other hand, $$|\Gamma(A_1\cup A_2)| = |\Gamma(A_1)\cup\Gamma(A_2)|$$
$$=|\Gamma(A_1)|+|\Gamma(A_2)|-|\Gamma(A_1\cap A_2)|$$
$$\leq |A_1|+|A_2|-|A_1\cap A_2|-1$$
$$= |A_1\cup A_2|-1. \quad (\rightarrow\leftarrow).$$ $\blacksquare$

(○) Can you find another proof?