

Theorem 11 ($V(G) = \{v_1, v_2, \dots, v_n\}$)

The vertices of a connected graph G can be enumerated so that $G_i =_{\text{def}} \langle \{v_1, v_2, \dots, v_i\} \rangle_G$ is connected for every $i = 1, 2, \dots, n$.

Proof. We construct all G_i 's recursively. Define $\langle \{v_1\} \rangle_G$ be G_1 , where v_1 is an arbitrary vertex of G . Assume that G_1, G_2, \dots, G_i have been constructed for $i \geq 1$ and $i < n$. Now, let $v \in V(G) \setminus V(G_i)$.

Since G is connected, there exists a path connecting v and v_1 .

Let v_{i+1} be the last vertex on the path not in G_i . (See Figure 13)

Then, $\{v_1, v_2, \dots, v_i, v_{i+1}\}$ induces a connected graph, G_{i+1} . \square

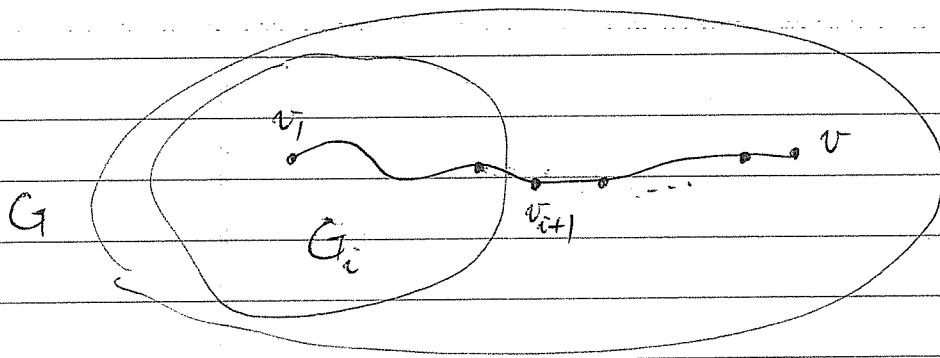


Figure 13. Graph enumeration

(*) The vertices of a tree^T can be enumerated such that every v_{i+1} has a unique neighbor in $\{v_1, v_2, \dots, v_i\}$. ($v \sim v_1$ is unique.)

Theorem 12

If T is a tree and G is any graph with $\delta(G) \geq \|T\|$, then

$$T \leq G.$$

(1st)

Proof. By induction on $\|T\|$. Clearly, the assertion is true for

$\|T\| = 1$. Assume that the assertion is true for $\|T\| = k \geq 2$. Consider a tree T with

$\|T\| = k+1$. Since $\delta(G) \geq k+1 > k$, G contains a tree (arbitrarily

given) of size k . (Let $T' = T - v_0$ where v_0 is a pendant vertex

of T , moreover $v_0 v' \in E(T)$.) Now, consider T' in G . v'

corresponds to a vertex u in $V(G)$. Since $\delta(G) \geq k+1$, u has

a neighbor u_0 which is not a vertex of T' . Note here that T' has

k vertices not including u . By attaching u_0 to T' , we have

a subgraph T . ▀

→ Direct construction

(2nd) Proof. Let T be a tree defined on $\{v_1, v_2, \dots, v_n\}$ such that

for each i , every vertex v_{i+1} has a unique neighbor in $\langle \{v_1, v_2, \dots, v_i\} \rangle$.

Now, we can construct directly on G the tree T by following the

enumerating order. (Pick the vertices and edges recursively.)

Since $\delta(G) \geq \|T\|$, there is always a vertex in G available

for the next choice. ▣

18'

Theorem 13

Every graph G with at least one edge has an ^(induced) subgraph

with $\delta(H) > \varepsilon(H) \geq \varepsilon(G)$.

Proof. We construct a sequence $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots$ of

induced subgraphs of G as follows. If G_i has a vertex v_i

of degree $\deg_{G_i}(v_i) \leq \varepsilon(G_i)$, let $G_{i+1} \stackrel{\text{def}}{=} G_i - v_i$; if not,

we terminate our sequence and set $H \stackrel{\text{def}}{=} G_i$.

Now, by the selection process, $\delta(H) > \varepsilon(H)$. Furthermore,

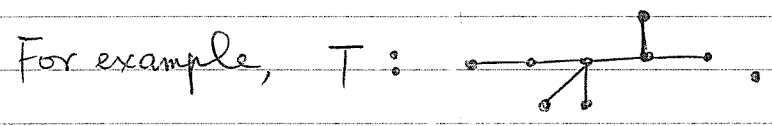
$$\begin{aligned} \varepsilon(G_{i+1}) &\geq \frac{\|G_{i+1}\|}{|G_i| - 1} = \frac{\|G_i\| - \deg_{G_i}(v_i)}{|G_i| - 1} > \frac{\|G_i\| - \varepsilon(G_i)}{|G_i| - 1} \\ &= \frac{\|G_i\| - \|G_i\|/|G_i|}{|G_i| - 1} = \frac{\|G_i\| (1 - \frac{1}{|G_i|})}{|G_i| - 1} = \frac{\|G_i\|}{|G_i|} = \varepsilon(G_i). \end{aligned}$$

Hence, $\varepsilon(H) \geq \varepsilon(G)$. ▣

(*) If there exists no v_i to delete, then G is the subgraph we need.

Tree Packing problem

Given a tree T of size $\|T\| = n$. Can we pack $2n+1$ copies of T into the complete graph of order $2n+1$?



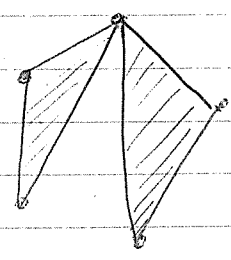
Fact 1 K_{2n+1} has $(2n+1) \cdot n$ edges.

Fact 2 If we can pack T ($\|T\| = n$) into K_{2n+1} , then it is a decomposition of K_{2n+1} into T 's.

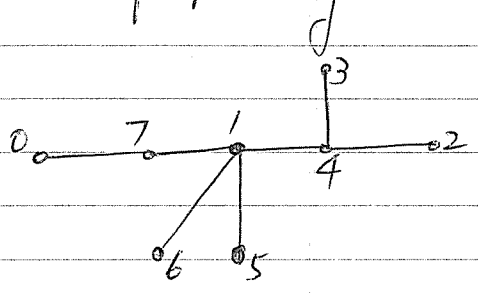
Review An \mathcal{H} -decomposition of G is a partition of $E(G)$ such that each part induces a graph in \mathcal{H} . If $\mathcal{H} = \{H\}$, then we simply say G has an H -decomposition.

Review A packing or \mathcal{H} -packing of G is a collection of edge-disjoint subsets of $E(G)$ such that each part induces a graph in \mathcal{H} .

e.g. A K_3 -packing of K_5 .

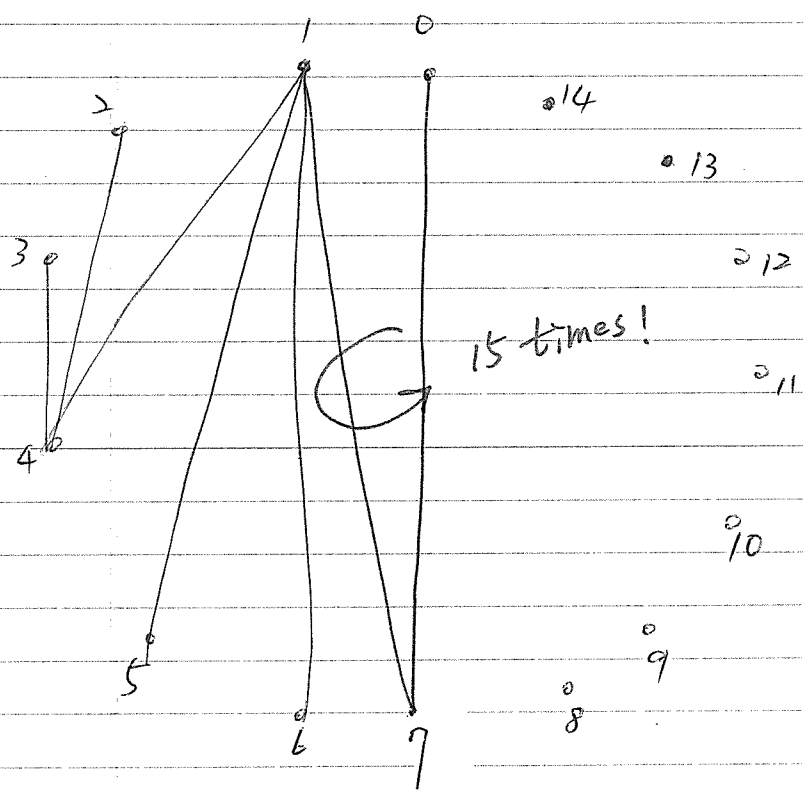


An idea of packing.



(Find a good labeling!)

[Graceful labeling or β -labeling]



15 times!

Definition (Graceful labeling) (or a β -labeling)

A vertex-labeling $\varphi: V(G) \rightarrow \{0, 1, 2, \dots, |G|\}$ is a graceful labeling if all the weights of uv , $|\varphi(u) - \varphi(v)|$ are distinct.

Graceful labeling tree conjecture: Trees are graceful! (G750,-)

$\kappa(G)$: Connectivity of G , $\kappa'(G)$: edge-connectivity of G

Theorem 14

For each connected graph G , $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

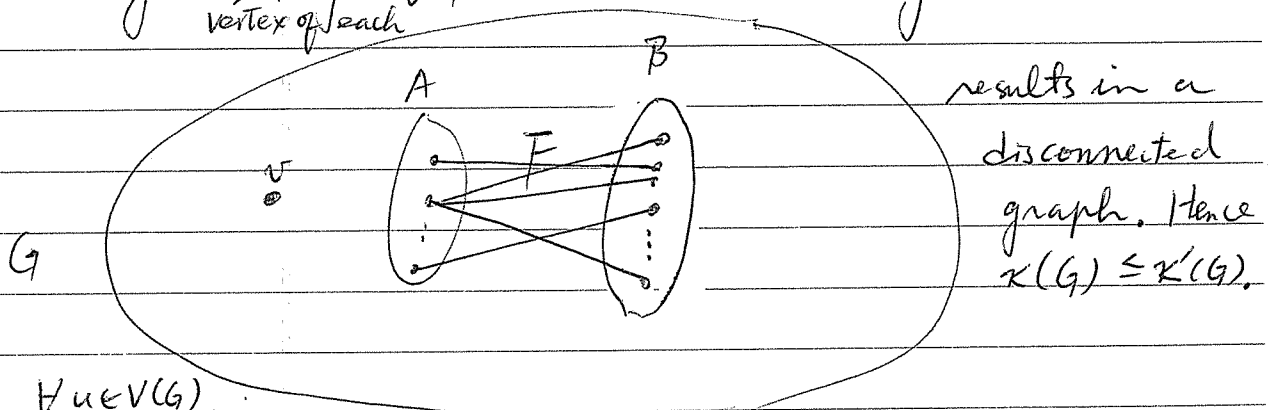
Proof.

② Let v be a vertex of G such that $\deg_G(v) = \delta(G)$. Since the deletion of all edges incident v will provide a disconnected graph, $\delta(G) \geq \kappa'(G)$.

① Let F be an edge cut of G which has $\kappa'(G)$ edges.
(Figure 14)

Case 1. $\exists v \in V(G)$, v is not incident any edge of F .

Let C be the component ^{in $G-F$} which contains v . Then, C contains exactly one edge of F . (?) Hence, deleting all such vertices _{vertex of each}



Case 2. $\forall u \in V(G)$, u is incident to an edge of F . (next page)

Figure 14. Edge-cut F

(*) It is interesting to know the graphs G satisfying $\kappa(G) = \delta(G)$.

Let C be the component of $G - F$ which contains an arbitrary vertex v . Now, if $wv \notin F$, then w is incident to an edge of F . Then, $\deg_G(v) \leq |F|$. This implies (again) $\kappa(G) \leq |F| = \kappa'(G)$. On the other hand all, $wv \in F$, $w \in N_G(v)$, then we have a complete graph of order $|F| + 1 = |G|$. $\kappa(G) = |G| - 1 = |F| = \kappa'(G)$. \square
(W.L.O.G.)

Theorem 15 (Mader, 1972)

Let $k \in \mathbb{N}$. Every graph G with $d(G) \geq 4k$ has a $(k+1)$ -connected subgraph H such that $\varepsilon(H) > \varepsilon(G) - k$.

Proof. For convenience, let $\varepsilon = \varepsilon(G)$ and consider $G' \leq G$ s.t.

$|G'| \geq 2k$ and $\|G'\| > \varepsilon \cdot (|G'| - k)$. Since $\|G\| = \varepsilon \cdot |G| > \varepsilon \cdot (|G| - k)$,

such graphs G' do exist. Let H be the one with minimum order.

Clearly, $|H| > 2k$, for otherwise, $\|H\| > \varepsilon \cdot k \geq 2k^2 > \binom{|H|}{2} = k \cdot (2k-1) \rightarrow \varepsilon$

The minimality of H implies that $\delta(H) > \varepsilon$, by Theorem 13 (we can choose a proper induced subgraph). Hence, $|H| \geq \varepsilon$. By the choice of

H , we have $\|H\| > \varepsilon(|H| - k)$ mentioned above, thus $\varepsilon(H) > \varepsilon - k$.

$$\left(\frac{\|H\|}{|H|} > \frac{\varepsilon(|H|) - \varepsilon k}{|H|} = \varepsilon - \frac{\varepsilon k}{|H|} \geq \varepsilon - \frac{|H| \cdot k}{|H|} = \varepsilon - k \right)$$

Therefore, H satisfies the condition we need. It's left to check that H is $(k+1)$ -connected. Suppose not. Let K be a set of k vertices in H such that $H - K$ is disconnected, see Figure 15.

Now, let $V(H) = U_1 \cup U_2$ such that $U_1 \cap U_2 = K$, $H_1 = \langle U_1 \rangle_H$, $H_2 = \langle U_2 \rangle_H$ and \wedge no vertices of $U_1 \setminus U_2$ is adjacent to vertices in $U_2 \setminus U_1$.

Let $v \in U_1 \setminus U_2$. Since $\wedge \deg_H(v) \geq \delta(H) > \varepsilon$, this implies that

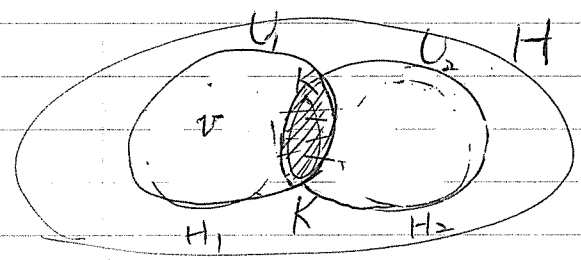
$|H_1| \geq \varepsilon \geq 2k$, so is $|H_2| \geq 2k$. Because of the fact that H is

the choice of G' with minimum order, $\|H_1\| \leq \varepsilon(|H_1| - k)$ and

$\|H_2\| \leq \varepsilon(|H_2| - k)$. Hence $\|H\| \leq \|H_1\| + \|H_2\| \leq \varepsilon(|H_1| + |H_2| - 2k) = \varepsilon(|H| - k)$.

A contradiction to

the choice of H .



$$U_1 \cap U_2 = K$$

Figure 15, $H_1 = \langle U_1 \rangle_H$ and $H_2 = \langle U_2 \rangle_H$

