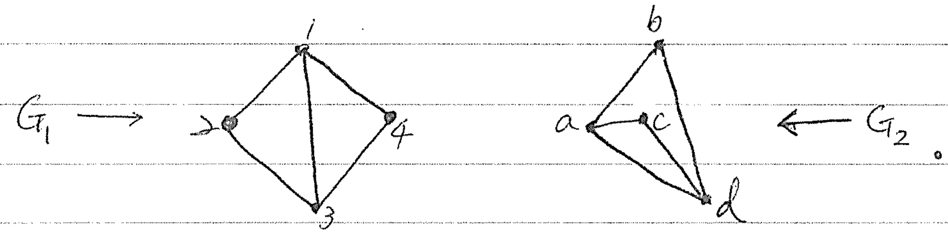
FGT Lecture 2 Sep. 19-21

Graph Isomorphism for Simple Graphs

Given two graphs, for example the following:



Are they "the same" or not ?

Fact If $V(G_1) \neq V(G_2)$ or $E(G_1) \neq E(G_2)$, then $G_1 \neq G_2$.

Observation

(1) G_1 and G_2 are of the same order, i.e., $|G_1| = |G_2|$.

(2) G_1 and G_2 are of the same size, i.e., $\|G_1\| = \|G_2\|$.

(3) G_1 and G_2 are of the same "incidence" structure:

$1 \longrightarrow a$, $3 \longrightarrow d$, $2 \longrightarrow c$, $4 \longrightarrow b$.

Review: Two sets A and B are of the same cardinality if there exists a bijection between A and B, i.e., $\exists$ $f: A \xrightarrow[\text{onto}]{1-1} B$. (A and B can be infinite sets.)

Definition (Isomorphic Graphs)

Two graphs G_1 and G_2 are isomorphic if there exists a bijection $f: V(G_1) \longrightarrow V(G_2)$ such that

$$\underset{\substack{\parallel\!\!\!\mid \\ uv \in E(G_1)}}{u \sim_{G_1} v} \Leftrightarrow \underset{\substack{\parallel\!\!\!\mid \\ f(u)f(v) \in E(G_2)}}{f(u) \sim_{G_2} f(v)}.$$

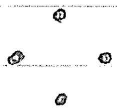
We use $G_1 \cong G_2$ to denote the fact that G_1 is isomorphic to G_2.

Fact $G_1 \cong G_2 \Rightarrow \begin{cases} ① \ |G_1| = |G_2| \\ ② \ \|G_1\| = \|G_2\| \\ ③ \ \text{Many others} \cdots. \\ \quad (\text{properties}) \end{cases}$

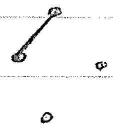
Fact Two graphs are non-isomorphic if they are not isomorphic.

Fact Isomorphism is an equivalence relation defined on $\mathcal{G} \times \mathcal{G}$ where $\mathcal{G}$ is the set of all simple graphs (graphs).

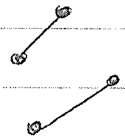
Example: Non-isomorphic graphs of order 4 (simple)



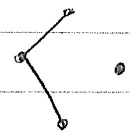
O_4



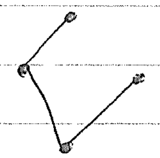
$O_2 \cup K_2$



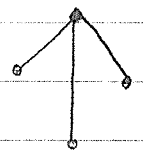
M_2



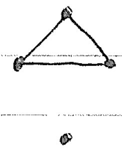
$P_3 \cup O_1$



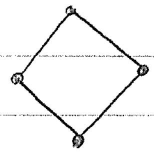
P_4



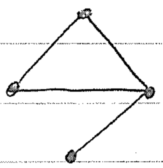
S_3

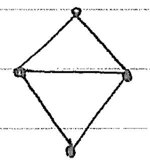


$K_3 \cup O_1$ or $C_3 \cup O_1$

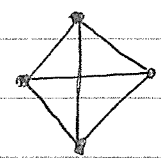


C_4



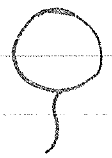


$K_4 - e$



K_4

(



Tadpole)

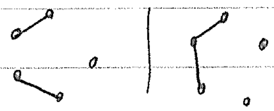
Homework: (How many non-isomorphic graphs are of order 6?)

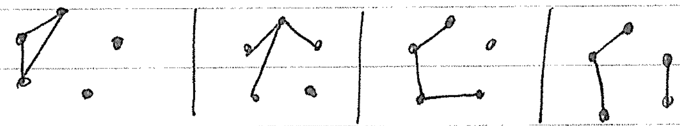
There are 34 non-isomorphic graphs of order 5.
(simple)

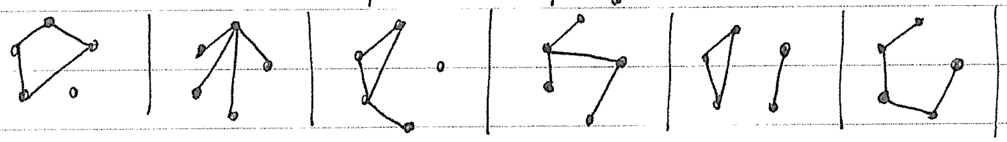
<u>Idea</u> First, we use the number of edges to partition the set of graphs of order 5.

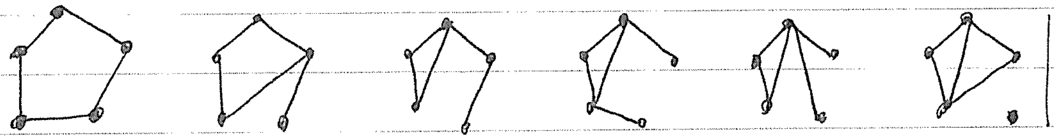
1. $e=0$,  1

2. $e=1$  1

3. $e=2$  2

4. $e=3$  4

5. $e=4$  6

6. $e=5$  6

7. $e=6 \approx e=4$ 6

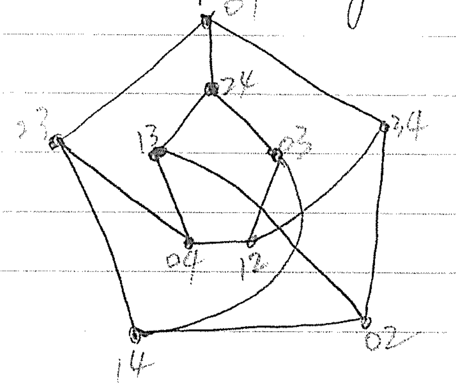
8. $e=7 \approx e=3$ 4

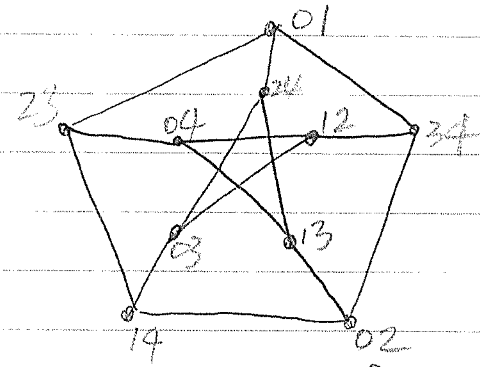
9. $e=8 \approx e=2$ 2

10. $e=9 \approx e=1$ 1

11. $e=10 \approx e=0$ 1

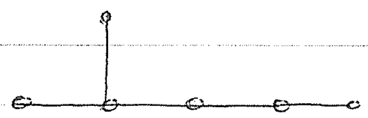
Are the following two graphs isomorphic?

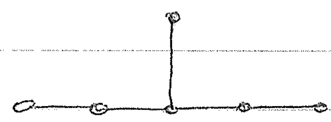




Yes! Petersen Graph

Are the following two graphs isomorphic?





No!

<u>Fact</u> Let $V = \binom{\mathbb{Z}_5}{2} = \{\text{2-element subsets of } \mathbb{Z}_5\}$

$= \{01, 02, 03, 04, 12, 13, 14, 23, 24, 34\}$

and $E = \{AB \mid A, B \in V \text{ and } A \cap B = \phi\}$. The (V, E) is isomorphic to Petersen graph. (<u>Kneser Graphs!</u>)

<u>Famous Graphs</u>

(!) So, how to determine whether two graphs are isomorphic or not. (Isomorphism Disease)

Possible approach

Definition (Degree sequence)

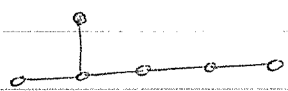
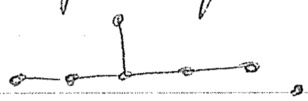
Let G be a graph and $V(G)=\{v_i \mid i=1,2,\cdots,n\}$. Let $d_i=\deg_G(v_i)$ and $\overset{\Delta(G)}{\overset{\|}{d_1}} \geq d_2 \geq \cdots \geq \overset{\delta(G)}{\overset{\|}{d_n}}$. Then, $\boxed{\langle d_1, d_2, \cdots, d_n\rangle \text{ is the degree sequence of } G.}$

Review 1. $\forall\ v\in V(G)$, $N(v)=\{u\in V(G) \mid uv\in E(G)\}$ and $N[v]=N(v)\cup\{v\}$. They are the neighborhood of v and closed neighborhood of v respectively.

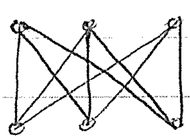
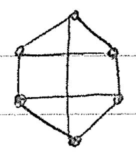
2. $|N(v)|$ is known as the $\boxed{\text{degree}}$ of v, denoted by $\deg_G(v)$. If the graph has a loop passing v, then this loop contributes two degrees.

3. A vertex is even (or odd) if its degree is even (or odd).

Example 1. $\langle 3,2,2,1,1,1\rangle$ is the degree sequence of both

 and .

2. $\langle 3,3,3,3,3,3\rangle$ is the degree sequence of both

 and .

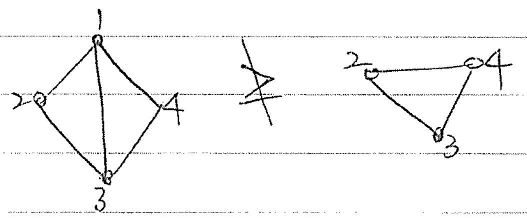
Can we say something about isomorphic graphs and degree sequences?

Conclusion (Trivial)

Two graphs may have the same degree sequence, but they are not isomorphic.

(Fact) If two graphs are isomorphic, then they have the same degree sequence.

Subgraphs

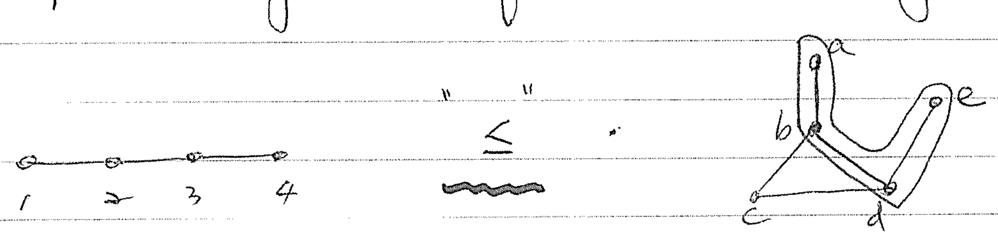


Definition (Strong sense)

Let G and H be two graphs such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Then H is a subgraph of G.

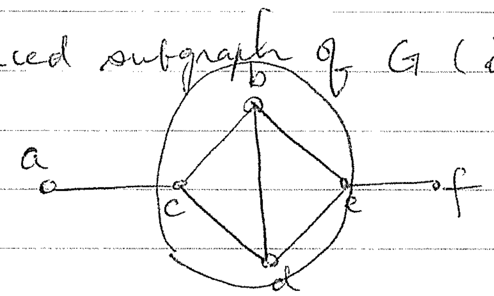
Definition (General version)

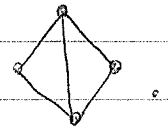
H is a subgraph of G if H is isomorphic to a subgraph (strong sense) of G, denoted by $H \leq G$.



Definition (Induced subgraphs)

Let G be a graph and $S \subseteq V(G)$ be a non-empty subset. The graph induced (or generated) by S, $\langle S \rangle_G$, is a subgraph of G such that for any two vertices u and v in S $u \sim_{\langle S \rangle_G} v$ if and only if $u \sim_G v$. $\langle S \rangle_G$ is called an induced subgraph of G (induced by S).

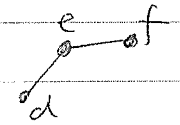
G: 

Let $S = \{b, c, d, e\}$. $\langle S \rangle_G$:  .

$\langle S \rangle_G \leq G$.
$\uparrow$ induced subgraph

Definition (Edge-induced subgraphs)

Let G be a graph and $\tilde{E} \subseteq E(G)$. Then the edge-induced subgraph of $\tilde{E}$, $\langle \tilde{E} \rangle_G$, is the subgraph of G whose edge set is $\tilde{E}$ and vertex-set is the set of vertices in G which are incident to an edge in $\tilde{E}$.

For example, let $\tilde{E} = \{de, ef\}$, then $\langle \tilde{E} \rangle_G$: 

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$\boxed{\text{(Fact)}}$ If two graphs G_1 and G_2 are isomorphic, then H is a subgraph of G_1 if and only if H is a subgraph of G_2.

Definition (H-free graphs).

A graph G is said to be H-free if H is not a subgraph of G.

Example. Petersen graph is C_3-free and also C_4-free.

Review. A cycle with $n \geq 3$ vertices is denoted by C_n.

Definition (Extremal graphs)

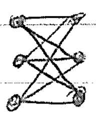
Consider all graphs $(\mathbb{G}_n)$ on n vertices. Let

$ex(n;H) = \max\{\|G\| \mid G \in \mathbb{G}_n \text{ and } G \text{ is } H\text{-free}\}$. The graphs G of order n and size $ex(n;H)$ are known as extremal graphs which forbids H.

Example. $n=6$, $ex(6;C_3) = 9$. (why?)

Note Two steps to show.

Step 1. There exists a graph of order 6 and size 9 which is C_3-free.



($ex(6;C_3) \geq 9$)

Step 2. All graphs of order 6 and size "≥ 10" contain a C_3.
(Harder step!) ($ex(6;C_3) \leq 9$)

(*) Prove this theorem

Theorem (Mantel, 1907)
Every graph of order n and size greater than $\lfloor \frac{n^2}{4} \rfloor$ contains a K_3. 2 - 8

Since $K_3 \nsubseteq G$, for every $xy \in E(G)$, $N_G(x) \cap N_G(y) = \phi$. This implies that $\deg_G(x) + \deg_G(y) \le |G| = n$. (Figure 1) Now, consider

$$\sum_{xy \in E(G)} (\deg_G(x) + \deg_G(y)) = \sum_{x \in V(G)} (\deg_G(x))^2 \quad \text{(Two-way counting)}.$$

$$\le n \cdot \|G\| = n \cdot e(G)$$

By Cauchy's inequality, $(2e(G))^2 = \left(\sum_{x \in V(G)} \deg_G(x)\right)^2 \le n \cdot \sum_{x \in V(G)} (\deg_G(x))^2$

$$\le n^2 \cdot e(G).$$

Hence, $e(G) \le \frac{n^2}{4}$. ■

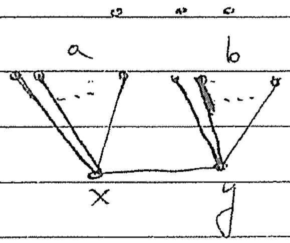


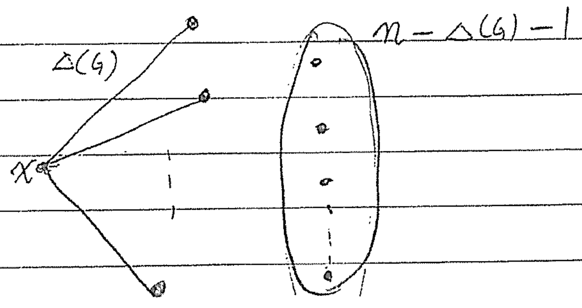
Figure 1. $\deg_G(x) = a+1$, $\deg_G(y) = b+1$

(2nd proof.)

Let $x \in V(G)$ be a major vertex, i.e., $\deg_G(x) = \Delta(G)$. (Figure 2)

Since $K_3 \nsubseteq G$, $\langle N_G(x) \rangle_G$ induces an empty graph. This implies that $\|G\| \le \Delta(G) + \Delta(G) \cdot (n - \Delta(G) - 1) = \Delta(G) \cdot (n - \Delta(G))$.

$\|G\|$ will take a maximum when $\Delta(G) = \lfloor \frac{n}{2} \rfloor$. Hence, we have the proof. ■



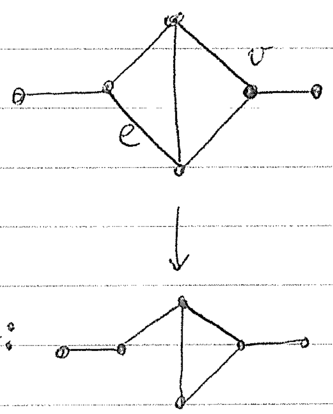
Problem $G \nsupseteq C_4$
$\|G\| = 14$
Find $\max\{\|G\|\}$.
(Bonus)

<u>Remark</u> To determine $ex(n; H)$ for a given H is a very difficult problem in general. But, for some special H, say $K_k, P_k, \cdots$, $ex(n; H)$ can be determined.

<u>Definition</u> (Deletion of vertices (and/or) edges)

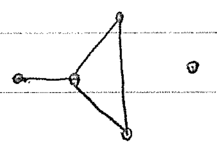
Let $v \in V(G)$. The graph $G-v$ is obtained by deleting the vertex v and all the edges incident to v. The graph $G-e$ where $e \in E(G)$ is a subgraph of G with the deletion of e from $E(G)$.

G:



$G-e$:

$G-v$:



Let $S \subseteq V(G)$ and $T \subseteq E(G)$. $G-S$ and $G-T$ can be defined accordingly. (Take away vertices (or edges) one by one.)

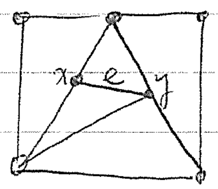
<u>Definition</u> (Graph minors)

A graph M is called a minor of G if M can be obtained from G by contracting edges, deleting vertices and edges.

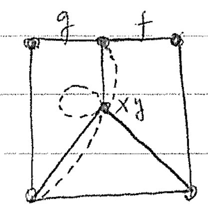
<u>Review</u> (Edge-contraction)

Given an edge $\underset{}{xy} = e$ of a graph G, the graph G/e is obtained from G by contracting e; that is to identify the vertices x and y and deleting resulting loops and duplicate edges.

G :



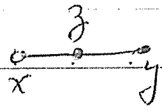
G/e :



<u>Example</u> K_4 is a minor of the above G.
(Contracting e, f and g.)

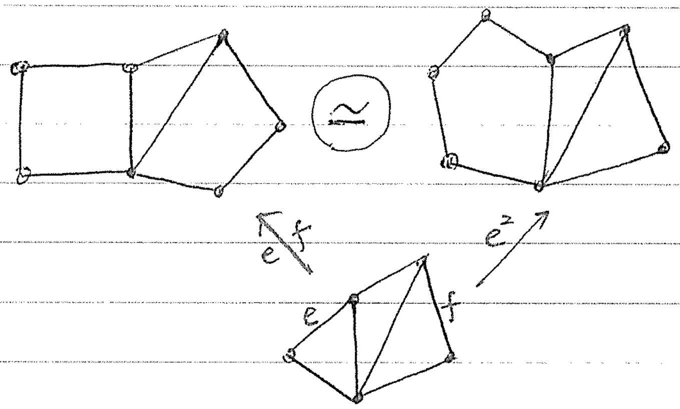
<u>Definition</u> (Subdivision)

A subdivision of an edge xy is obtained by adding a new vertex z such that we have edges xz and zy.



<u>Definition</u> (Homeomorphic Graphs)

Two graphs are homeomorphic if they can be obtained by subdividing edges (consecutively) of a fixed graph.



(These two graphs are "topologically" the same.)

<u>Remark</u>. Two cycles are homeomorphic.