Theorem 5.5 (BEST Theorem)

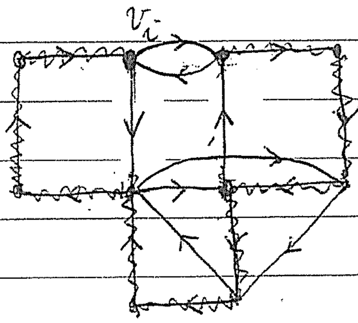
A digraph D has an eulerian (directed) circuit if and only if D is strongly connected and for each vertex $v \in V(D)$, $\deg^+_D(v) = \deg^-_D(v)$. Moreover, if D is an eulerian graph, $s(D)$ is the number of distinct eulerian circuits, then

$$s(D) = t_i(D) \cdot \prod_{j=1}^{n} (\deg^+_D(v_j) - 1)! \quad \text{for every } i \in \{1, 2, \cdots, n\}$$

where $t_i(D)$ is the number of spanning trees oriented toward v_i.

(Note: The counting part of the theorem was proved by (independently) two groups de Bruijn and van Aardenne-Ehrenfest, Smith and Tutte.)

Proof. The existence part can be obtained by a similar argument as the "multigraph" version.

D: 

$s(D) = 1 \cdot 1 = 1$

Figure 6. Spanning tree oriented toward v_i

Theorem 6

Every graph G contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G)+1$ (provided $\delta(G) \geq 2$).

Proof. Let $\langle v_0, v_1, \cdots, v_\ell \rangle$ be a longest path. Then, $N_G(v_\ell) \subseteq \{v_0, v_1, \cdots, v_{\ell-1}\}$. For otherwise, we have a longer path. Since v_ℓ has at least $\delta(G)$ neighbors $\ell \geq \deg_G(v_\ell) \geq \delta(G)$. This concludes the first part. Now, let i be the smallest index in $\{0, 1, 2, \cdots, \ell-1\}$ such that $v_i v_\ell \in E(G)$. Hence, $(v_i, v_{i+1}, \cdots, v_\ell)$ is a cycle of length at least $\delta(G)+1$. ■

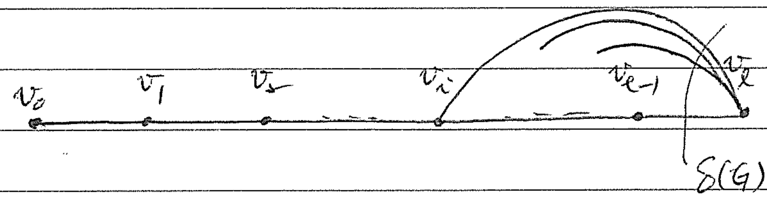


Figure 7. $\ell - i \geq \delta(G)$.

Exercise A-1 Every connected graph G contains a path or cycle of length at least $\min\{2\delta(G), |G|\}$.

Theorem 7

For each connected graph G, $\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\,\mathrm{rad}(G)$.

Proof. It suffices to consider the second inequality. Let u and v be two vertices in G such that $d(u,v) = diam(G)$. Let w be a vertex in the center of G, i.e, $ecc(w) = rad(G)$. By the fact that "d" is a metric, $d(u,w) + d(w,v) \geq d(u,v)$. This implies that

$ecc(w) + ecc(w) = 2\,rad(G) \geq d(u,w) + d(w,v) \geq d(u,v) = diam(G)$. ▨

(Note. The eccentricity of $w \in V(G)$ is $\max\{d(x,w) \mid x \in V(G)\}$.)

Theorem 8

A graph of minimum degree δ and girth g (shortest cycle) has at least $n_0(\delta, g)$ vertices where

$$n_0(\delta, g) = \begin{cases} 1 + \delta \sum_{i=0}^{r-1} (\delta-1)^i, & \text{if } g \underset{def}{=} 2r+1 \text{; and.} \\ 2 \cdot \sum_{i=0}^{r-1} (\delta-1)^i, & \text{if } g = 2r. \end{cases}$$

Proof.

Case 1, $g = 2r+1$, $r \geq 1$.

Let v_0 be a fixed vertex in G, see Figure 8.

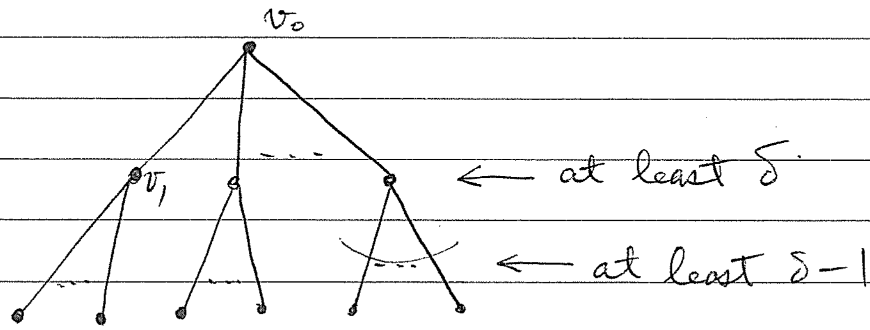


Figure 9

Then, there are at least δ neighbors of v_0, and for each neighbor, say v_1, v_1 has at least $\delta-1$ neighbors. Since $g=2r+1$, G contains at least $1+\delta+\delta(\delta-1)+\cdots+\delta(\delta-1)^{r-1}$ vertices. This concludes the proof of the Case 1.

Case 2. $g=2r$

In this case, we start with an edge u_0v_0, see Figure 10. By a similar argument, G contains at least $2\cdot[(\delta-1)+(\delta-1)^2+\cdots+(\delta-1)^{r-1}]$ vertices. ∎

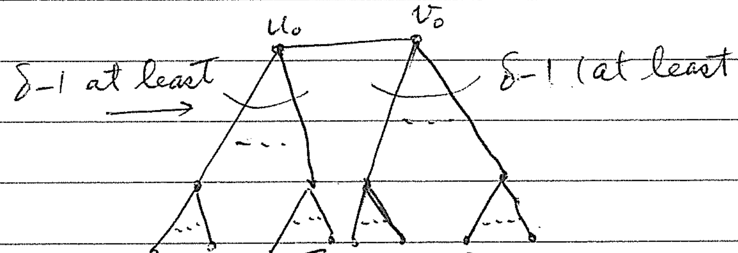


Figure 10.

Theorem 9

If $\delta(G) \ge 3$, then $g(G) < 2\log_2 |G|$.

Proof. Note that if $\delta_1 \ge \delta_2 \ge 3$, then $n_0(\delta_1, g) \ge n_0(\delta_2, g)$.

It suffices to consider $n_0(3,g)$. By Theorem 8,

$$|G| \ge n_0(3,g) = 2^r + 2^r - 2 > 2^r \quad (g=2r) \text{ and}$$

$$|G| \ge n_0(3,g) = 1 + 3\cdot\frac{2^r - 1}{2-1} = \frac{3}{\sqrt{2}}\, 2^{\frac{g}{2}} - 2 > 2^{\frac{g}{2}} \quad (g = 2r+1).$$

This implies that $r < \log_2 |G|$ and thus $g < 2\log_2 |G|$. ∎

$\boxed{(d,g)\text{-graph and } (d;g,h)\text{-graph}} \rightarrow 12'$

Theorem 10 A (d,g)-cage is a d-regular graph with girth g and minimum number of vertices. Prove that A (d,g)-cage is 2-connected. $(g \ge 3)$

Proof. First, we claim that if $g_1 > g_2$, then A (d, g_1)-cage contains more vertices than the the order of a (d, g_2)-cage.

Suppose not. Let G_1 and G_2 be two cages respectively and $|G_1| \le |G_2|$.

It suffices to consider the case $g_1 = g_2 + 1$. Let $\|G_1\| = f(d; g_1)$ and $\|G_2\| = f(d, g_2)$.

A-2: Find as many (d,g)-cages as possible.

Review

(•) The girth of a graph G, $g(G)$, is the size of a smallest cycle in G. If G contains no cycle, then $g(G) =_{def} +\infty$.

(•) The perimeter of a graph G, $pm(G)$, is the largest size of a cycle in G. Clearly, $pm(G) \le |G|$ and the equality holds when G has a Hamilton cycle (hamiltonian cycle).

(•) A (d,g)-graph is a d-regular graph with $g(G) = g$.

(•) A (d,g)-cage is a (d,g)-graph with minimum order.

(*) To determine whether a graph contains a cycle of length $3 \le k \le |G|$ is very difficult in the sense of algorithms.

(a) d is even

Let C be a cycle (in G_1) of length g_1 and $uv_1, uv_2 \in E(C)$, moreover $N_{G_1}(u) = \{v_1, v_2, \dots, v_d\}$. Let $E' = \{v_1v_2, v_3v_4, \dots, v_{d-1}v_d\}$. Now, consider $G_1 - u + E'$ and denote the component contains v_1 by G_1'. Clearly, G' is a simple graph and G_1' contains a cycle of length g_2: $C - u + v_1v_2$. Further, if C' is a cycle of G_1' and $E(C') \cap E' = \phi$, then C' is a cycle of G_1 and thus of length at least g_1. On the other hand, if $E(C') \cap E' \neq \phi$, then let v_iv_j be one of the edges. Let P be a $\langle v_i, \dots, v_j \rangle$ path on C' satisfying $E(P) \cap E' = \phi$. So, $P + \{uv_i, uv_j\}$ is a cycle of G_1. This implies that $|E(C')| \geq g_1 - 1 = g_2$. This concludes that G_1' is a d-regular graph with girth at least g_2 and $|G_2| \leq |G_1'| = f(d; g_1) - 1 \leq |G_1|$.

(b) d is odd

Let C be a cycle (in G_1) of length g_1 and $uv_1, uv_2 \in E(C)$. Let $N_{G_1}(u) = \{v_1, v_2, \dots, v_{d-1}, w\}$. Clearly, $w \notin V(C)$. For otherwise, we have a cycle of length less than g_1. (See Figure 11.)

Now, let $N_{G_1}(w) = \{u, x_1, x_2, \cdots, x_{d-1}\}$ and G_1' be the component (contains v_1) of $G - \{u, w\} + \{v_{2i-1}v_{2i},\ x_{2i-1}x_{2i} \mid 1 \le i \le (d-1)/2\}$.

Again, G_1' is simple and G_1' is a (d, g)-graph with at most $f(d; g_1) - 2$ vertices.

Hence, $f(d; g_2) < f(d, g_1)$.

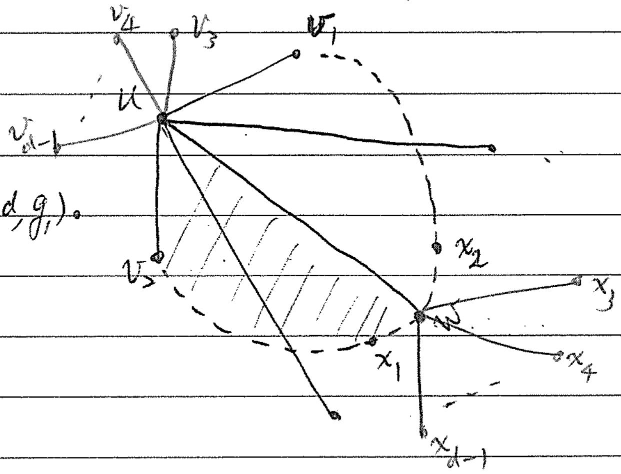


Figure 11, Shorter cycle

Proof of the theorem (A (d, g)-cage is 2-connected.)

Suppose not. Let u be a cut-vertex. Let $C_1, C_2, \cdots, C_w$ be the components of $G - u$, with $|V(C_i)| \le |V(C_{i+1})|$, $i = 1, 2, \cdots, w-1$.

Consider C_1. In C_1, $\forall\ v_1, v_2 \in V(C_1) \cap N_G(u)$, $d(v_1, v_2) \ge g - 2$. (Figure 12) Let C' be an isomorphic copy of C_1 with isomorphism φ. Now, construct a new graph H where $V(H) = V(C') \cup V(C_1)$ and $E(H) = E(C') \cup E(C_1) \cup \{v\varphi(v) \mid v \in V(C_1) \cap N_G(u)\}$. By observation, $|H| < |G|$, H is d-regular and H has girth at least $\min\{g, 2g-2\} = g$.

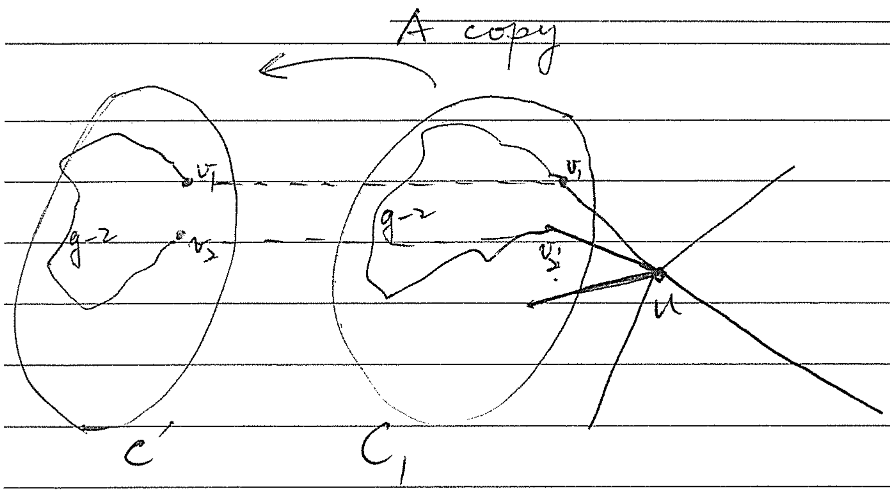


Figure 12, Construction of H.

This implies that G is not a (d,g)-cage, a contradiction. ■

Facts

1. It has been proved that a $(3,g)$-cage is 3-connected.

2. It is conjectured that a (d,g)-cage is d-connected.

Reference

H. L. Fu, K. C. Huang and C. A. Rodger, Connectivity of cages, JGT, Vol. 24, No. 2, 187–191 (1997).

Connectivity of Cages

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ABSTRACT

A $(k;g)$-graph is a k-regular graph with girth g. Let $f(k;g)$ be the smallest integer ν such there exists a $(k;g)$-graph with ν vertices. A $(k;g)$-cage is a $(k;g)$-graph with $f(k;g)$ vertices. In this paper we prove that the cages are monotonic in that $f(k;g_1) < f(k;g_2)$ for all $k \geq 3$ and $3 \leq g_1 < g_2$. We use this to prove that $(k;g)$-cages are 2-connected, and if $k = 3$ then their connectivity is k.

1. INTRODUCTION

All graphs in this note are simple. The length of a shortest odd or even cycle in a graph G is called the *odd girth* or the *even girth* of G, respectively. Throughout this paper let $g = g(G)$ denote the smaller of the odd and even girths of G (so g is the *girth* of G), and let $h = h(G)$ denote the larger; then the *girth pair* of G is defined to be (g, h). A k-regular graph with girth pair (g, h) is called a $(k; g, h)$-graph. For any $k \geq 1$ and any $g \not\equiv h \pmod 2$ with $3 \leq g < h$, let $f(k; g, h)$

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denote the smallest integer ν such that there exists a $(k; g, h)$-graph with ν vertices. Similarly, a k-regular graph with girth g is called a $(k; g)$-graph, and let $f(k; g)$ denote the smallest integer ν such that there exists a $(k; g)$-graph with ν vertices; a $(k; g)$-graph with $f(k; g)$ vertices is called a *cage*. Cages have been studied widely since introduced by Tutte in 1947 [3]; see [4] for a survey referring to 70 publications.

Several interesting questions concerning girth pairs of graphs remain open. For example, it is clear that $f(k; g) \leq f(k; g, h)$, and this inequality may be strict; for example, the $(k; 4)$-cage is $K_{k,k}$ [4], so contains no 5-cycles, so in this case $f(k; 4) < f(k; 4, 5)$. Related to this observation is a conjecture of Harary and Kovacs [2] who believe that if g is odd then $f(k; g) = f(k; g, g+1)$. But whether $f(k; g, h) \leq f(k; h)$ remains unknown. Harary and Kovacs proved [2] that $f(k; h-1, h) \leq f(k; h)$. They also conjectured that all $(k; g, h)$-graphs of order $f(k; g, h)$ are 2-connected. In this paper we prove the related conjecture that cages are 2-connected. Our proofs rely on knowing that cages are monotonic in the sense that $f(k; g_1) < f(k; g_2)$ for all $g_1 < g_2$. While this may be known to some, we can find no reference to the result, so a proof is included here. For any undefined terminology, see [1].

2. MONOTONICITY AND CONNECTIVITY OF CAGES

There have been many papers that find bounds on $f(k; g)$ (see [4] for a survey). We begin by considering $f(k; g)$, proving that cages are monotonic, a result that will also be of use in considering the connectivity of cages.

Theorem 1. *For all $k \geq 3$ and $3 \leq g_1 < g_2, f(k; g_1) < f(k; g_2)$.*

Proof. It suffices to show that if $k, g \geq 3$ then $f(k; g) < f(k; g+1)$. So let G be a $(k; g+1)$-graph with $f(k; g+1)$ vertices.

Suppose k is even. Let C be a cycle of length $g+1$ in G containing the edges uv_1 and uv_2. Let $N_G(u) = \{v_1, \ldots, v_k\}$ be the neighborhood of u in G, and let $E' = \{v_1v_2, v_3v_4, \ldots, v_{k-1}v_k\}$. Let G' be the component of $G - u + E'$ that contains v_1. Since $g + 1 \geq 4, N_G(u)$ is an independent set of G, so $E' \cap E(G) = \emptyset$, and so G' is a simple graph. Clearly G' contains the cycle $(C - u) + v_1v_2$ of length g. Also, if C' is a cycle in G' then: if $E' \cap E(C') = \emptyset$ then C' is a cycle in G; and if $E' \cap E(C') \neq \emptyset$ then let P be a (v_i, v_j)-path that is a subgraph of C' with $E(P) \cap E' = \emptyset$, so $P + \{uv_i, uv_j\}$ is a cycle in G, so C' has length at least g (since C' contains P and at least one edge in E'). So G' has no cycles of length less than g, and is therefore a $(k; g)$-graph with at most $f(k; g+1) - 1$ vertices, so $f(k; g) < f(k; g+1)$.

Suppose k is odd. Let C be a cycle of length $g+1$ in G containing uv_1 and uv_2. Let $N_G(u) = \{v_1, \ldots, v_{k-1}, w\}$. Clearly $w \notin V(C)$, for if C is the cycle $(u, v_2, \ldots, x_1, w, x_2, \ldots, v_1)$ then $(u, v_2, \ldots, x_1, w)$ is a cycle of length less than the girth of G. Let $N_G(w) = \{x_1, \ldots, x_{k-1}, u\}$. Let G' be the component of $(G - \{u, w\}) + \{v_{2i-1}v_{2i}, x_{2i-1}x_{2i} | 1 \leq i \leq (k-1)/2\}$ that contains v_1. Since $g + 1 \geq 4, N_G(u)$ and $N_G(w)$ are independent sets of G, so G' is simple. Clearly $C - u + v_1v_2$ is a cycle in G' of length g, and (as in the previous case) no cycle in G' has length less than g. Therefore G' is a $(k; g)$ graph with at most $f(k; g+1) - 2$ vertices, so $f(k; g) < f(k; g+1)$. ∎

We can now use Theorem 1 to prove the following result.

Theorem 2. *All $(k; g)$-cages are 2-connected.*

Proof. Suppose that G is a connected (k, g)-graph that contains a cut vertex u. Let $C_1, \ldots, C_w$ be the components of $G - u$, with $|V(C_i)| \leq |V(C_j)|$ for $1 \leq i < j \leq w$. Clearly

$$d_{C_1}(v_1, v_2) \geq g - 2 \qquad \text{for all } v_1, v_2 \in V(C_1) \cap N_G(u). \tag{1}$$

Let C' be a copy of C_1 with $V(C') \cap V(C_1) = \emptyset$, and let f be an isomorphism between C_1 and C'. Form a new graph from the union of C_1 and C' by joining each $v \in V(C_1) \cap N_G(u)$ to $f(v)$ with an edge.

Clearly H is k-regular, and has fewer vertices than G (since $|V(C')| \leq |V(C_2)|$ and $u \notin V(H)$). Also, by (1), any cycle in H containing an edge $vf(v)$ has length at least $2(g-2)+2 = 2g-2$, so H has girth at least $\min\{g, 2g-2\} = g$. Therefore by Theorem 1, G is not a $(k; g)$-cage, and the result follows. ∎

3. FURTHER RESULTS

While it is good to know that cages are 2-connected, we believe that their connectivity is much higher. Indeed, we are bold enough to make the following conjecture.

Conjecture. *All simple $(k; g)$-cages are k-connected.*

In support of this conjecture, we now prove the following result.

Theorem 3. *All cubic cages are 3-connected.*

Proof. Suppose G' is a $(3; g)$-cage. By Theorem 2, G' has connectivity at least 2. Suppose G' has connectivity 2. The following construction of a graph G is depicted in Figure 1.

Since G' is a cubic graph, G' has an edge-cut consisting of two edges, say e and f. Let H' and W' be the two components of $G' - \{e, f\}$, let $e = x_0y_0$ and $f = x_1y_1$, where $\{x_0, x_1\} \subseteq H'$ and $\{y_0, y_1\} \subseteq W'$. Let $d_{W'}(y_0, y_1) = d \leq d_{H'}(x_0, x_1) = D$. Let $P = (w_0 = y_0, w_1, w_2, \ldots, w_d = y_1)$ be a shortest (y_0, y_1)-path in W', let $Q' = (h_0 = x_0, h_1, h_2, \ldots, h_D = x_1)$ be a shortest (x_0, x_1)-path in H' and let $Q = (h_0, h_1, \ldots, h_{d-1})$ be the (x_0, h_{d-1})-subpath of Q'. For each $i \in \{0, 1\}$ let z_i be the unique neighbor of y_i in W' that is not in P. Let R be the path $(z_0, x_0, w_1, h_1, w_2, h_2, \ldots, w_{d-1}, h_{d-1})$. Let $H = H' - E(Q)$ and let $W = (W' - E(P)) - \{y_0, y_1\}$. Let $G = (H \cup W \cup R) + \{x_1z_1\}$ (see Fig. 1).

Clearly G is a cubic graph with $|V(G')| - 2$ vertices. We now show that G has girth at least g, so the result will then follow from Theorem 1 which will contradict G' being a $(3; g)$-cage.

Any cycle in G that is also in G' clearly has length at least g. Any cycle in G that is not in G' contains at least two edges in $E(R) \cup \{x_1z_1\}$; let C be a cycle containing exactly two such edges, say e_1 and e_2. We consider several cases.

Case 1. Suppose $e_1 = x_0z_0$ and $e_2 = h_{i-1}w_i$ or h_iw_i with $1 \leq i \leq d-1$.

Let P_1 be a shortest (z_0, w_i)-path in W. Then P_1 is a path in W'. Let P_2 be the (y_0, w_i)-subpath of P; so P_2 has length i. Then clearly $(P_1 \cup P_2) + y_0z_0$ contains a cycle of length at most $i + 1 + d_W(z_0, w_i)$. Since $(P_1 \cup P_2) + y_0z_0$ is a subgraph of G', $i + 1 + d_W(z_0, w_i) \geq g$. For each $l \in \{i-1, i\}$, $d_H(x_0, h_l) \geq d_{H'}(x_0, h_l) = i - 1$, so C has length at least $d_H(x_0, h_l) + d_W(z_0, w_i) + 2 \geq i - 1 + g - (i+1) + 2 = g$.

Case 2. Suppose $e_1 = x_0z_0$ and $e_2 = x_1z_1$.

Let P_1 be a shortest (z_0, z_1)-path in W. Then $(P_1 \cup P) + \{y_0z_0, y_1z_1\}$ contains a cycle, and this cycle has length at most $d + 2 + d_W(z_0, z_1)$. Since this cycle is also a subgraph of

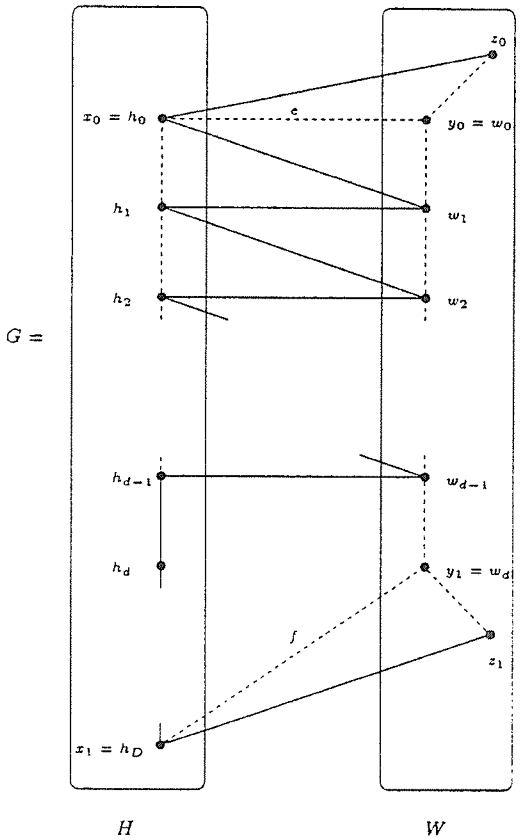

FIGURE 1. Dashed lines are edges in G' not in G.

$G', d+2+d_W(z_0, z_1) \geq g$. Clearly $d_H(x_0, x_1) \geq d_{H'}(x_0, x_1) = D$. Therefore C has length at least $d_H(x_0, x_1) + d_W(z_0, z_1) + 2 \geq D + g - (d+2) + 2 \geq g$.

Case 3. Suppose $e_1 = h_{i-1}w_i$ or $h_i w_i$ and $e_2 = h_{j-1}w_j$ or $h_j w_j$, with $1 \leq i \leq j \leq d-1$.

If $i = j$ then we can assume $e_1 = h_{i-1}w_i$ and $e_2 = h_i w_i$, so $C - \{e_1, e_2\} + h_{i-1}h_i$ is a cycle in G', and so has length at least g. Therefore C has length at least $g+1$.

If $i < j$ then let P_1 be a shortest (w_i, w_j)-path in W. Since $P_1 + \{w_l w_{l+1} | i \leq l < j\}$ contains a cycle in G', P_1 has length at least $g - (j-i)$. Also, for each $l_1 \in \{i-1, i\}$ and each $l_2 \in \{j-1, j\}, d_H(h_{l_1}, h_{l_2}) \geq d_{H'}(h_i, h_{j-1}) = j - 1 - i$. So C has length at least $g - (j-i) + (j-1-i) + 2 = g+1$.

Case 4. Suppose $e_1 = h_{i-1}w_i$ or $h_i w_i$ with $1 \leq i \leq d-1$ and $e_2 = x_1 z_1$.

As in the previous case $d_W(w_i, z) \geq g - (d+1-i)$, and for each $l \in \{i-1, i\}$ $d_H(h_l, x_1) \geq d_{H'}(h_i, x_1) = d - i$. Therefore C has length at least $g - (d+1-i) + (d-i) + 2 = g+1$.

Thus in every case, if C contains exactly two edges in R then C has length at least g. If C contains more than two edges in R then it follows even more easily that C has length at least g, so the result is proved. ∎

ACKNOWLEDGMENTS

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