Theorem 1 – 5 (Lecture 1) Sep. 12~14

Theorem 1 (Veblen, 1912)

The edge set of a graph can be partitioned into cycles if and only if every vertex has even degree.

Proof. ($\Rightarrow$) A vertex contained in t cycles has degree $2t$.
($\Leftarrow$) The cycles can be obtained recursively. We start with finding the first cycle. Let $\langle x_0, x_1, \cdots, x_l \rangle$ be a path of maximal length l in G. Since $x_0x_1 \in E(G)$, $\deg_G(x_0) \geq 2$. Let $y (\neq x_1)$ be a neighbor of x_0, i.e., $x_0y \in E(G)$. Now, $y \in \{x_2, x_3, \cdots, x_l\}$. For otherwise, we have a longer path. So, if $y = x_i$, then we have a cycle $C = (x_0, x_1, \cdots, x_i)$. The process continues in $G - E(C)$. (Each vertex is of even degree in the graph $G - E(C)$.) ∎

Theorem 2 (Mantel, 1907)

Every graph of order n and size greater than $\lfloor \frac{n^2}{4} \rfloor$ contains a triangle (C_3 or K_3). Proof. (1st)

Since $K_3 \nsubseteq G$, for every $xy \in E(G)$, $N_G(x) \cap N_G(y) = \phi$. This implies that $\underline{\deg_G(x) + \deg_G(y) \le |G| = n}$. (Figure 1) Now, consider

$$\sum_{xy \in E(G)} (\deg_G(x) + \deg_G(y)) = \sum_{x \in V(G)} (\deg_G(x))^2 \quad \text{(Two-way counting)}.$$

$$\le n \cdot \|G\| = n \cdot e(G)$$

By Cauchy's inequality, $(2e(G))^2 = \left(\sum_{x \in V(G)} \deg_G(x)\right)^2 \le n \cdot \sum_{x \in V(G)} (\deg_G(x))^2$

$$\le n^2 \cdot e(G).$$

Hence, $e(G) \le \frac{n^2}{4}$. ■

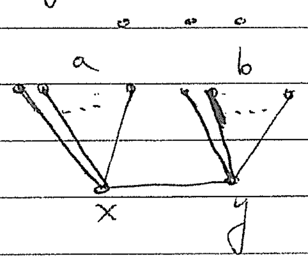


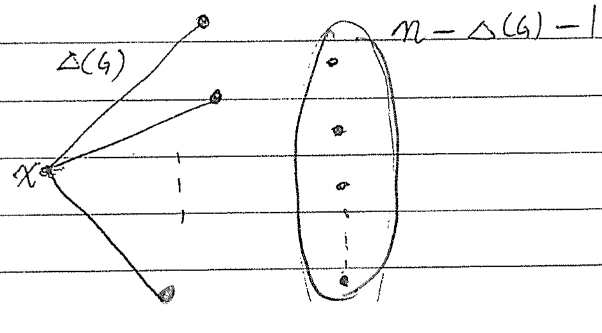
Figure 1. $\deg_G(x) = a+1$, $\deg_G(y) = b+1$

(2nd proof.)

Let $x \in V(G)$ be a major vertex, i.e., $\deg_G(x) = \Delta(G)$. (Figure 2)

Since $K_3 \nsubseteq G$, $\langle N_G(x) \rangle_G$ induces an empty graph. This implies that $\|G\| \le \Delta(G) + \Delta(G) \cdot (n - \Delta(G) - 1) = \Delta(G) \cdot (n - \Delta(G))$.

$\|G\|$ will take a maximum when $\Delta(G) = \lfloor \frac{n}{2} \rfloor$. Hence, we have the proof. ■

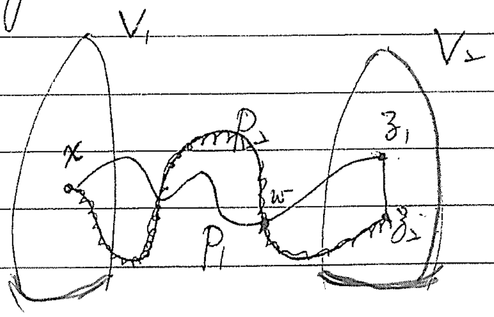


Theorem 3 A graph is bipartite if and only if it does not contain an odd cycle.

Proof. ($\Rightarrow$) Let $G = (A, B)$ where A and B are its partite sets. If $(x_0, x_1, \cdots, x_\ell)$ is a cycle of G, then x_0 and x_ℓ must in different partite sets. Hence, the index ℓ must be odd, thus the cycle is of even length.

($\Leftarrow$) W.L.O.G., let G be a connected graph. Let $x \in V(G)$ and $V_1 = \{y \mid y \in V(G)$ and $d(x,y)$ is even$\}$. Hence, $x \in V_1$. Let $V_2 = V(G) \setminus V_1$. It suffices to claim that both V_1 and V_2 are independent sets. First, consider V_2. Clearly, for each $z \in V_2$, $d(x,z)$ is odd. Suppose that $z_1, z_2 \in V_2$ and $z_1 \sim z_2$ ($z_1z_2 \in E(G)$). (Figure 3) Let P_1 and P_2 be the two paths such that $P_1 = \langle x, \cdots, z_1 \rangle$ and $P_2 = \langle x, \cdots, z_2 \rangle$, moreover they are the shortest paths connecting x to z_1 and x to z_2 respectively.

Figure 3



Let w be the last vertex in which P_1 and P_2 intersect. Also, let $\|P_1\| = 2s+1$ and $\|P_2\| = 2t+1$. (Note that if $V(P_1) \cap V(P_2) = \{x\}$, then we have an odd cycle (x, P_1, z_1, z_2, P_2) (length $2s+2t+3$).) Now, if w does exist, then $\langle x, P_1, w\rangle$ and $\langle x, P_2, w\rangle$ are of the same length, let the length be h. (?) So, the cycle $(w, \cdots, z_1, z_2, \cdots)$ is of length $(2s+1-h)+(2t+1-h)+1 = 2s+2t-2h+3$, an odd integer. Thus, an odd cycle exists, a contradiction. Hence, V_2 is an independent set. A similar argument can be applied to show that V_1 is also an independent set. (x is not adjacent to any vertex of $V_1 \setminus \{x\}$.) Figure 4.

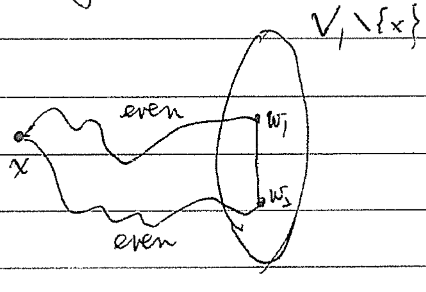


Figure 4.

<u>Open problem</u> How many edges can a graph G of order n have such that $G \not\supseteq C_4$?

Theorem 4 The following statements are equivalent for a graph G.

(a) G is a tree. (G is connected and acyclic.)

(b) G is connected and every edge of G is a bridge.

(c) G is a maximal acyclic graph. (If x and y are not adjacent, then $G+xy$ contains a cycle.

Proof. (a) $\Rightarrow$ (b)

Let xy be an edge of G and $G-xy$ is connected. Then, there exists a path P connecting x and y in $G-xy$. Clearly, G contains a cycle (x, P, y) in G, a contradiction.

(b) $\Rightarrow$ (c) If G is not acycle, then a cycle edge is not a bridge. Hence, G is acyclic.

If G is not a maximal acyclic graph, then there exists a pair of vertices z_1 and z_2 in G such that $G+z_1z_2$ is also acyclic. Since G is connected, there exists a path connecting z_1 and z_2, say P. This implies that (z_1, P, z_2) is a cycle in $G+z_1z_2$, a contradiction.

(c) $\Rightarrow$ (a) If G is not connected, then there exists a pair of vertices x_1 and x_2 such that $G+x_1x_2$ is also acyclic. (in different components)

x_1 and x_2

Theorem 5 (Euler, 1941)

A nontrivial connected graph (multigraph) has an eulerian circuit (Euler circuit) if and only if each vertex has even degree. Moreover, a connected graph has an eulerian trail from a vertex x to a vertex $y \neq x$ if and only if x and y are the only two vertices of odd degree.

Proof. The second statement follows directly from the first one. We prove the first statement.

($\Rightarrow$) If a circuit passes a vertex x h times, then $\deg_G(x) = 2h$.

($\Leftarrow$) By induction on $\|G\|$. Since $\|G\| \geq 1$, $\delta(G) \geq 2$ and thus G contains a cycle. (G is not a tree!)

Let Z be a circuit in G with the maximum number of edges. If Z is an eulerian circuit, then we are done. Suppose not. Let H be a nontrivial component of $G - E(Z)$. Since G is connected, $V(H) \cap V(Z) \neq \emptyset$. Let $x \in V(H) \cap V(Z)$. Now, H is a nontrivial connected graph (even graph). Hence, H contains an eulerian circuit Y (by induction). By using x, we can attach

Z and Y together to obtain a larger circuit. (Figure 5) This contradicts to the maximality of $|E(Z)|$. Hence, Z must be an eulerian circuit of G. ▨

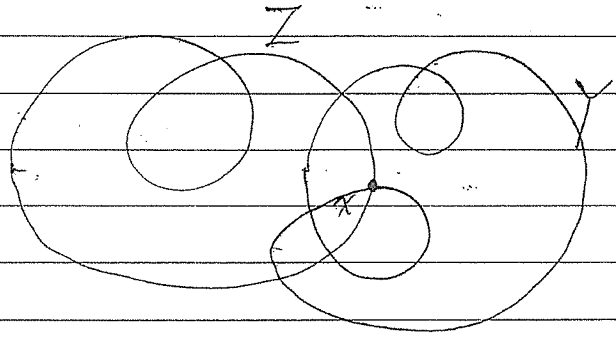
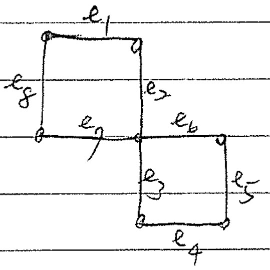


Figure 5. Attaching Z and Y.

<u>Open problem</u>

Find the number of distinct eulerian circuits of an eulerian graph G. (Two circuits are the same if they can be obtained each other by a cyclic shift of edges.)



$$\langle e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8 \rangle$$
$$=_{def} \langle e_3, e_4, e_5, e_6, e_7, e_8, e_1, e_2 \rangle$$
$$\neq_{def} \langle e_1, e_2, e_6, e_5, e_4, e_3, e_7, e_8 \rangle$$