7 Fundamental Graph Theory

Configurations of vertices (nodes) and connections occur in many applications of modern era. They are utilized to represent electrical circuits, roadways, wireless sensor networks, or organic molecules. In social sciences, they can also represent the relationship between people, parties and species. Therefore, as discrete mathematical models are concerned, “Graphs” play the most important role. In this chapter, we shall introduce some of the basic properties of graphs. For more details, the readers are encouraged to read a more comprehensive textbook of Graph Theory, for examples [1,2,3].

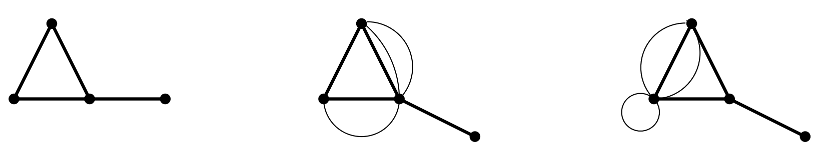
7.1 Graphs, digraphs and weighted graphs

The following definition may look quite different from what we have seen in other textbooks.

Definition 7.1.1. *Let V be a non-empty set and E be a collection of (multi)-subsets of V. Then, (V,E) is a graph defined on V, V and E are vertex-set and edge-set of G respectively.*

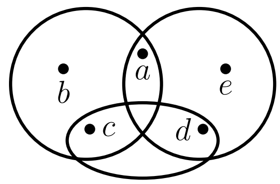
Let $G = (V,E) = (V(G), E(G))$ be a graph. Then, G is a *finite* (resp. infinite) *graph* if V is a finite (resp. infinite) set. G is a *simple graph* if E is a collection of 2-subsets (without repetition) of V. In case that 2-subset can occur multiple times, then G is called a *multigraph*, and the largest number of repetitions for an edge is called the multiplicity of the graph, denoted by $\mu(G)$.

If the size of an edge in a multigraph can be one, then the edge of size one is called a loop and the graph is a *pseudograph*. On the other hand, if the size of edge can be larger than two, than we have a *hypergraph*. The edges of a hypergraph is also named as a hyperedge. Figure 7.1 gives examples of different graphs.



7.1.(a) Simple graph 7.1.(b) Multigraph 7.1.(c) Pseudograph

Therefore, a graph can be viewed as a subset system of a given set V. This fact reveals that we can use a (0,1)-matrix to represent a graph $G = (V,E)$.

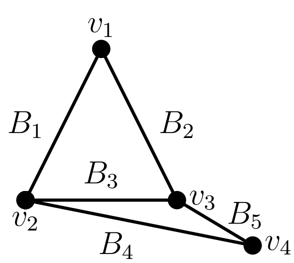


7.1.(d) Hypergraph with $E = \{\{a,b,c\},\{c,d\},\{a,d,e\}\}$

This matrix is known as an incidence matrix of a graph G, denoted by $B(G)$. See Fig.7.1.2. for an example.

Definition 7.1.2. *Let $G = (V, E)$ where $V = \{v_1, v_2, \dots, v_p\}$ and $E = \{B_1, B_2, \dots, B_q\}$. Then, the incidence matrix of G, $B(G)$, is a $p \times q$ (0,1)-matrix where $B(G)(i,j) = 1$ if $v_i \in B_j$ and $B(G)(i,j) = 0$ otherwise.*

G:



$B(G)$:

	B_1	B_2	B_3	B_4	B_5
v_1	1	1	0	0	0
v_2	1	0	1	1	0
v_3	0	1	1	0	1
v_4	0	0	0	1	1

Fig. 7.1.2. An example of incidence matrix

Observe that if G contains a loop, then one of the columns has only one "1", and a column has more than two 1's provided the graph is a hypergraph.

Two vertices of a graph G are adjacent if they belong to an edge. In case that these two vertices are in fact the same one, then we have a loop. We can also use a matrix to represent a graph by way of adjacency.

Definition 7.1.3. *Let $G = (V, E)$ where $V = \{v_1, v_2, \dots, v_p\}$. Then the adjacency matrix of G, $A(G)$, is a $p \times p$ (0,1)-matrix such that $A(G)(i,j) = 1$ if v_i and v_j are adjacent, and $A(G)(i,j) = 0$ otherwise. For convenience, we use v_iv_j to denote the edge $\{v_i, v_j\}$.*

Consider the graph in Figure 7.1.2, $A(G)$ is as follows,

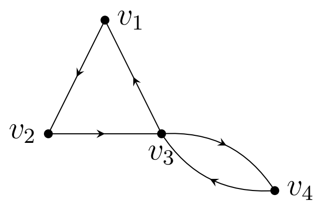
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Fig. 7.1.3. An example of adjacency matrix

In modeling a network, the relationships between two vertices u and v does impose directions. For example, u is adjacent to v but not the other direction. Therefore, the following notion is natural.

Definition 7.1.4. *Let V be a non-empty set and A be a set of ordered pairs obtained from $V \times V$. Then, $D = (V, A)$ is a directed graph, digraph in short.*

For convenience in depicting a digraph, we put an arrow sign on an arc(ordered pairs) (u, v) where u is a tail and v is a head, see Figure 7.1.4 for an example.



$$v = \{v_1, v_2, v_3, v_4\}$$
$$A = \big\{(v_1, v_2), (v_2, v_3), (v_3, v_1), (v_3, v_4), (v_4, v_3)\big\}$$

Figure 7.1.4. A digraph

Note that it is not easy to represent a digraph by using an incidence matrix, but we can use adjacency matrix in a similar set up.

Definition 7.1.5. *Let $D = (V, A)$ be a digraph with $V = \{v_1, v_2, \ldots, v_p\}$. Then, the adjacency matrix of D is a $p \times p$ (0,1)-matrix $A(D)$ such that $A(D)(i, j) = 1$ if (v_i, v_j) is an arc of A and $A(D)(i, j) = 0$ otherwise.*

So, for the digraph in Fig. 7.1.4, its adjacency matrix is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

Fig. 7.1.5. Adjacency matrix of a digraph

It is not difficult to see that the adjacency matrix of a graph is symmetric, but the adjacency matrix of a digraph may not be symmetric. If we consider a graph as a digraph, then an edge v_iv_j can be thought as the union of two arcs (v_i, v_j) and (v_j, v_i). In modeling a network, this idea is very common, since we can travel from v_i to v_j and v_j to v_i as well.

Another extension of a graph is the so-called *weighted graph* which can be extended from either graph or digraph. We consider the digraph version here.

Definition 7.1.6. *Let $D = (V, A)$ be a digraph. A weighted digraph D is a pair (D, w) where w is a mapping from A into $\mathbb{R}$. The weight of an arc (u, v) is denoted by $w(u, v)$.*

As a special type of weighted digrpah, we may use $D_k = \{0, 1, 2, \cdots, k\}$ to be the weight-set. Let $w^+(u)$ be the total weights of all arcs with tail u and $w^-(v)$ be the total weights of all arcs with head v. An assignment of weights on a digraph $D = (V, A)$ is called a *(k+1)-flow*, if for each arc in A, its weight is in D_k and for each $u \in V$, $w^+(u) = w^-(u)$.

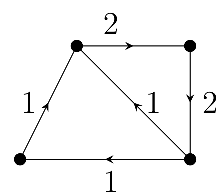
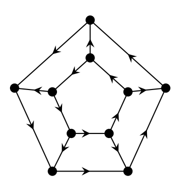


Figure 7.1.6. A 3-flow

Exercise 1. *Does the following graph has a 3-flow or a 4-flow?*



Assign a weight function on the edge set of a graph is also known as an edge-labeling of a graph. Many problems (with or without) applications can be modeled by using the labeling. By the way, we can also assign weights on vertices, known as vertex-labeling. Besides, we can also use signs $(+, -)$ to be the weights which is useful in social networks.

7.2 Subgraphs andd basic properties

For the rest of this section, we consider simple graphs. Two graphs with the same number of vertices and edges respectively can be drawn in many different ways, for example in Figure 7.1.7. So, it is important to tell whether they are essentially the same graph.

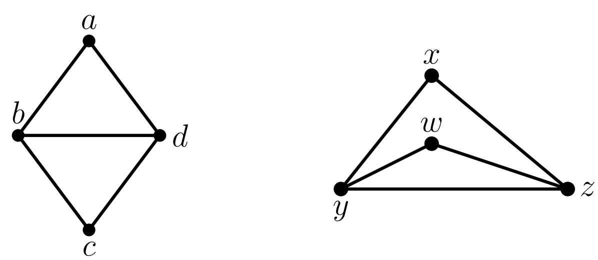


Fig 7.2.1. Two isomorphic graphs

Definition 7.2.1. *Two graphs* $G_1 = (V_1, E_1)$ *and* $G_2(V_2, E_2)$ *are isomorphic if there exists a bijection* φ *from* V_1 *onto* V_2 *such that* $u_1v_1 \in E_1$ *if and only if* $\varphi(u_1)\varphi(v_1) \in E_2$.

So, it is not difficult to see that the two graphs in Figure 7.2.1 are isomorphic. Also, the isomorphisms of graphs are indeed an equivalence relation. This observation allows us to count exactly how many non-isomorphic graphs of a given order, say 4, see Figure 7.2.2.

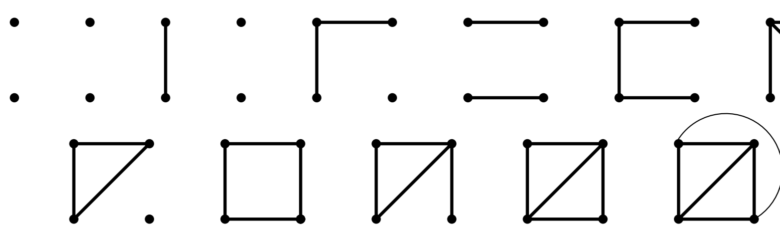


Figure 7.2.2. Non-isomorphic graphs of order 4

So, it is easy to notice that two isomorphic graphs must have the same order and size. But, given two graphs G_1 and G_2, to determine whether G_1 is isomorphic to G_2 is not easy at all. We may use the following notion to see how much two graphs are looking "the same"?

Definition 7.2.2. *Let* $G = (V, E)$ *be a graph. A graph* $H = (V', E')$ *is a subgraph of* G *if* $V' \subseteq V$ *and* $E' \subseteq E$.

A more general idea about subgraph can be defined as follows. We shall use this version throughout this chapter.

Definition 7.2.3. *A graph H is a subgraph of G, denoted by $H \leqslant G$, if H is isomorphic to a subgraph of G defined in Definition 7.2.2.*

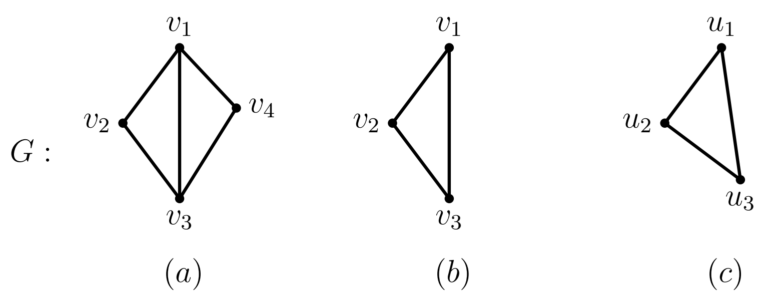


Figure 7.2.3. Graph in (c) is a subgraph of G.

A bit of reflection, aa subgraph of a graph G can be obtained by deleting vertices and edges. The following definition plays an important role.

Definition 7.2.4. *Let $G = (V, E)$ be a graph and v be a vertex in $V\big(V(G)\big)$. The set of vertices which are adjacent to v is called the* neighborhood *of v, denoted by $N_G(v)$ and $N_G(v) \cup \{v\}$ is called the closed neighborhood of v, denoted by $N_G[v]$. $|N_G(v)|$ is called the* degree *of v in G, denoted by $deg_G(v)$.*

In case that there is ambiguity in considering a graph G, we may simply use $N(v)$ or $N[v]$ instead. Now, we are ready for deleting vertices from a graph.

Definition 7.2.5. *Let $G = (V, E)$ be a graph and $S' \subseteq V$. Then the graph obtained by deleting the set of vertices S' from G is $G - S' =_{def} G'$ where $V(G') = V \backslash S'$ and $E' = E \backslash E''$. Here, E'' is the set of edges e in E such that e is incident to a vertex of S'.*

The following fact is known as the first theorem of Graph Theory.

Theorem 7.2.1. *Let $G = (V, E)$ be a graph. Then $\sum_{v \in V} deg_G(v) = 2|E|$.*

Proof. It follows by the fact that each edge contributes two degrees to the degree sum. □

Corollary 7.2.1. *The number of vertices with odd degree in a graph is even.*

Though this corollary is very simple to see, it plays an important role in many aspects. Interesting readers may refer to [3] for many applications.

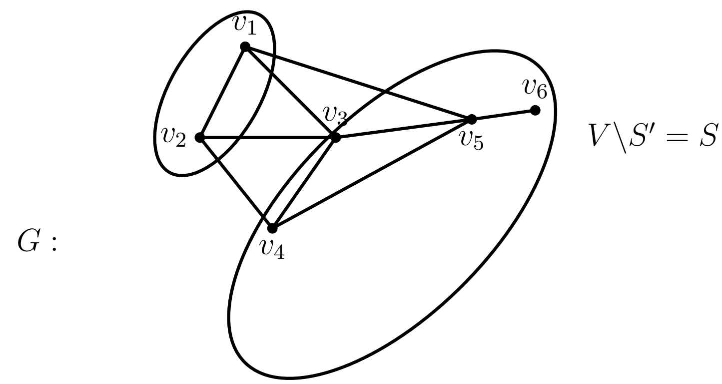


Figure 7.2.4. The graph $G - S$

The graph $G - S'$ is also known as an induced subgraph of G induced by $V \backslash S' =_{def} S$, denoted by ¡$S$¿$_G$. This fact reveals that an induced subgraph can be obtained by deleting vertices suitably. For example, the graph in Figure 7.2.4. can be obtained by deleting v_1 and v_2 consecutively.

Note that not all subgraphs of a graph are induced. The above definition shows that ¡S¿$_G$ contains all the edges in G whose two vertices (of an edge) are contained in S. By observation, if two graphs are isomorphic, then they contain isomorphic induced subgraphs. This shows that the following two graphs are not isomorphic (Figure 7.2.5.) since the Petersen graph does not have an induced subgraph C_4(4-cycle) in Graph (b).

(a) Petersen Graph (b) Generalized Petersen Graph

Figure 7.2.5. Two non-isomorphic graphs

Unfortunately, knowing proper induced subgraphs ¡S¿$_G$ of G where $S \subsetneq V(G)$ may not be able to know the graph G.

Ulam's reconstruction conjecture

Let $G = (V, E)$ where $V = \{v_1, v_2, \dots, v_p\}$. If we are given p graphs $\{G - v_i\}$, $i = 1, 2, \dots, p$, then G can be reconstructed up to isomorphism.

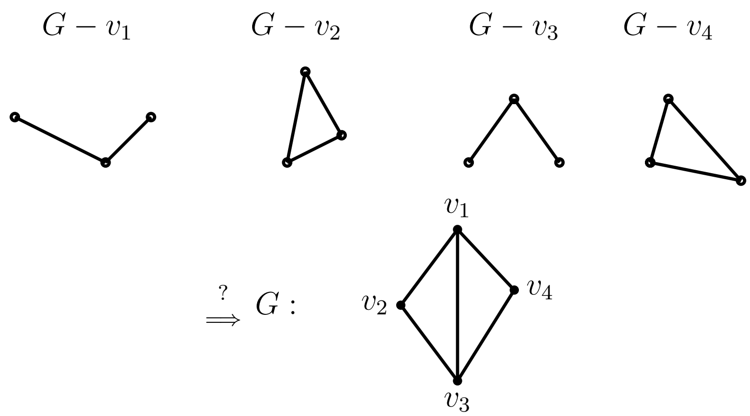


Figure 7.2.6. An example of reconstruction

This conjecture has been verified for quite a few classes of graphs, but in general it remains unsettled.

Definition 7.2.5. *A graph G is called reconstructible if G can be reconstructed (up to isomorphism) from $\{G - v\}_{v \in S}$ for a subset $S \subseteq V(G)$.*

Exercise 2. *A graph $G = (V, E)$ is a bipartite graph if V can be partitioned into two subsets V_1 and V_2 such that ¡V_1¿$_G$ and ¡V_2¿$_G$ contain no edges. Prove that if $|E| = 8$, then G contains an induced subgraph (V', E') such that $|E'| = 4$.*

Exercise 3. *Find a bipartite graph of size $2m$ such that there does not exist an induced subgraph of size m.*

Remark It was conjectured by *Chang* and *Hwang* in [4] that if G is a bipartite graph of size 2^k, then G contains an induced subgraph of size 2^{k-1}.

7.3 Connectivity of graphs

Let G be a graph with vertex-set $V = \{v_1, v_2, \dots, v_p\}$. A sequence of vertices in V, ¡$v_{i_1}, v_{i_2}, \dots, v_{i_t}$¿ is called a walk if for each $1 \le j \le t - 1$, $v_{i_j}, v_{i_{j+1}}$ is an edge of G. Clearly, a walk may contain repeated vertices and edges. The length of a walk is the number of edges in the walk. If $v_{i_1} = v_{i_t}$, then it's a walk closed.

A walk is called a trail if there do not exist repeated edges and a trail is a path if no vertices occurred more than once. A closed walk is a *circuit* if no edges repeated and it is a cycle if no vertices repeated.

For convenience, a walk of length k is denoted by W_k. For path and cycle of length k, they are denoted by P_{k+1} and C_k. The following theorem provides a tool to count the number of walks with length k.

Theorem 7.3.1. *Let $A_{p\times p}$ be the adjacency matrix of a graph G with vertex set $\{v_1, v_2, \ldots, v_p\}$. Then, the number of walks with length k is equal to $A^k(i, j)$.*

The proof can be obtained by induction on k. We leave it as an exercise.

Exercise 4. *Prove Theorem 7.3.1.*

Exercise 5. *Use Theorem 7.3.1. to find the number 3-cycles, 4-cycles and 5-cycles in a graph G.*

Definition 7.3.1. *A graph G is* connected *if for any two vertices u and v in $V(G)$ there exists a path connecting u and v.*

Definition 7.3.2. *A graph G is disconnected if it is not connected. A maximal connedted subgraph of G is called a component of G and the number of components of G is denoted by $c(G)$.*

Clearly, if G is connected, then $c(G) = 1$. On the other hand, if $c(G) \geq 2$, then G is disconnected.

The complement of a graph G, denoted by $\overline{G}$, is a graph defined on $V(G)$ such that two vertices u and v are adjacent if and only if u and v are not adjacent in G. Hence, if we let $G = (V, E)$ and $\overline{G} = (V, \overline{E})$, then the graph $(V, E \cup \overline{E})$ is in fact the complete graph of order $|V|$.

There are many good properties in discussing the relationship between G and $\overline{G}$. The following result is a very important fact.

Proposition 7.3.1. *If G is a disconnected graph, then $\overline{G}$ is a connected graph. As a consequence, either G or $\overline{G}$ is a connected graph.*

Proof. Since G is disconnected, $c(G) \geq 2$. Let the components of G be $Z_1, Z_2, \ldots, Z_t$, $t \geq 2$. Now, if two vertices of G are in distinct components, then there is an edge joining them. On the other hand, if they are in the same component, then we pick one vertex from the other component and connect these vertices by a path through that vertex. $\square$

If G is a connected graph, then we can find a path connecting given two vertices u and v. This motivates us to define the distance between u and v.

Definition 7.3.3. *The distance of two vertices u and v in a graph G, denoted by $d_G(u,v)$ (or $d(u,v)$ in short), is the length of a shortest path between u and v. If there exist no paths connecting u and v, then we let $d(u,v) = +\infty$.*

So, it is not difficult to see that the *distance* is in fact a metric, i.e., (1) $d(u,v) = 0$ iff $u = v$, (2) $d(u,v) = d(v,u)$, and (3) $d(u,v) + d(v,w) \geq d(u,w)$.

Definition 7.3.4. *Let G be a graph. Then the eccentricity of a vertex v, denoted by $e_G(v)$ (or $e(v)$ in short), is $max\{d(u,v)|u \in V(G)\}$. The diameter of G, $diam(G)$, is equal to $max\{e(v)|v \in V(G)\}$ and the radius of G, $rad(G)$, is equal to $min\{e(v)|v \in V(G)\}$. For each v, if $e(v) = rad(v)$, then v is in the center of G, $C(G)$.*

The diameter and radius of a graph satisfy the following inequalities.

Proposition 7.3.2. *For each graph G, $rad(G) \leq diam(G) \leq 2rad(G)$.*

Proof. The first inequality is trivial. We prove the second one. Let u and v be two vertices in G such that $d(u,v) = diam(G)$. Let $w \in C(G)$. By the metric inequality $d(u,v) \leq d(u,w) + d(w,v) = d(w,u) + d(w,v) \leq 2e(w) = 2rad(G)$. $\square$

"Distance" plays an important role in the study of graph structure and also has many applications related to using this notion. One of them which has been important in Chemistry is the Wiener Index.

Definition 7.3.5. *The Wiener index of a graph, $W(G)$, is defined as* $\sum_{u,v \in V(G)} d(u,v)$.

It is interesting to find $W(G)$ for a given G. Clearly, we would like to consider the case when G is connected.

So, if G is a graph of order n, how many edges do we need to make sure that G is connected. The following notion will be useful to consider such a problem.

Definition 7.3.6. *A graph G is* acyclic *if G contains no cycles. An acyclic connected graph is called a* tree *and an acyclic graph is called a* forest*.*

Theorem 7.3.2. *The following statements about graph T are equivalent.*

(1) T is a tree.

(2) T is connected and $||T|| = |T| - 1$.

(3) T is acyclic and $||T|| = |T| - 1$.

(4) T is connected and $||T||$ is minimum.

(5) For each edge $e \in E(T)$, $T - e$ is disconnected.

(6) For any two vertices there exists a unique path connecting them.

Proof. We prove (1) $\Rightarrow$ (2) here and leave the others for exercies. By the definition of tree, T is connected. So, we prove the relationship between the order and the size of T by using induction on the order of T. Clearly, the assertion is true when $|T| = 1$. Assume that it is true for trees of order n. Now, consider T where $|T| = n + 1$. Let u and v be two vertices of $V(T)$ such that $d(u, v) = diam(T)$. Then, both u and v are of degree 1. Suppose not. Let $deg(u) \geq 2$. Then, u has a neighbor w different from the one on the shortest path from v to u. First, if w is not on $u - v$ path, then we conclude that either $d(w, v) > diam(G)$ or there exists another $w - v$ path with length not greater than $diam(G)$. Both of them are not allowed since the later one will creat a cycle. On the other hand, if w is on the path, then we have a cycle as well. This contradicts to the definition of a tree. Therefore, we conclude that v is of degree 1. Now, consider $T - v$. This graph is connected and $||T - v|| = |T - v| - 1$ by induction hypothesis. Hence, $||T|| = |T| - 1$ follows. □

Exercise 6. *Let a and b be positive integers such that $a \leq b \leq 2a$. Construct a graph G satisfying $rad(G) = a$ and $diam(G) = b$.*

Exercise 7. *Prove Theorem 7.3.2.*

A graph H is called a *spanning* subgraph of G if $V(H) = V(G)$. In case that H is a tree, then H is known as a *spanning tree*. It is not difficult to notice the fact: Every connected graph contains a spanning tree. This can be achieved by deleting the so-called "cycle edges" if there are any. (An edge e is a cycle-edge of a graph G if there exists a cycle in G which contains e.)

Now, we are ready to consider the connectivity of graphs.

Definition 7.3.7. *A vertex subset S of a connected graph G is called a cut-set if $G-S$ is either disconnected or contains only one vertex. Similarly, an edge set T of a connected graph G is an edge-cut if $G-T$ is disconnected.*

It is interesting to know the minimum size of S and T respectively.

Definition 7.3.8. *The connectivity of a nontrivial connected graph G is defined as $\kappa(G) = min\{|S| \big| G-S$ is disconnected or $|G-S| = 1\}$. The edge-connectivity of G is $\kappa'(G) = min\{|T| \big| T \subseteq E(G)$ and $G-T$ is disconnected$\}$.*

Example $\kappa(T) = \kappa'(T) = 1$ if T is a tree and $\kappa(C_n) = \kappa'(C_n) = 2$ if $n \geq 3$.

Theorem 7.3.3. *Let $\delta(G)$ denote the minimum degree of a connected graph. Then, $\delta(G) \geq \kappa'(G) \geq \kappa(G)$.*

Proof. The first inequality is easy to see, we prove the second inequality. Let T be an edge-cut of G such that $|T| = \kappa'(G)$. Now, for each edge in T we choose one vertex and put it in the set S. Therefore $|S| \leq |T|$ since there may have different edges in T in which we choose the same vertex incident to them. By the definition of deleting vertices $G-S$ contains no edges no edges in T. Hence, $G-S$ is either disconnected or $G-S$ has only one vertex. Thus, S is a cut-set and we conclude the proof. □

Exercise 8. *Let a, b, c be three positive integers satisfying $a \leq b \leq c$. Construct a graph G such that $\kappa(G) = a$, $\kappa'(G) = b$ and $\delta(G) = c$.*

Exercise 9. *Prove that if $\kappa(G) = 3$, then for any three vertices in G, there exists a cycle containing these vertices.*

Exercise 10. *Prove that if $\kappa'(G) = 3$, then for any two vertices u and v, there exist three edge-disjoint paths connecting u and v.*

In general, we can only say $\kappa(G) \leq \delta(G)$. But, the equality does hold for certain graphs. For example, trees and complete graphs. As to $\kappa'(G)$, $\kappa'(G) = \delta(G)$ holds a very interesting class of graphs, diameter 2 graphs.

Theorem 7.3.4. *[13] If G is a graph of diameter 2, then $\kappa'(G) = \delta(G)$.*

Proof. Let G be a graph of order p and $p(H)$ be the order of a subgraph H of G. Let T be a set of edges of size $\kappa'(G)$ such that $G - T$ is disconnected. Since T is minimal, $G - T$ has exactly two components H_1 and H_2. For convenience in proof, let $p_1 = p(H_1) \leq p(H_2) = p_2$.

Since G is of diameter 2, either each vertex of H_1 is adjacent to some vertex of H_2 or each vertex of H_2 is adjacent to some vertex of H_1. Thus, $\kappa'(G) = |S| \geq min\{p(H_1), p(H_2)\} = p(H_1)$. (1)

Now, consider $u \in V(H_1)$. Let $d_i(u)$ denote the number of vertices in $H_i(i = 1, 2)$ adjacent to u in G. Then, by (1), $p(H_1) \leq \kappa'(G) \leq \delta(G) \leq deg(u) = d_1(u) + d_2(u) \leq \big(p(H_1) - 1\big) + d_2(u)$. (2)

Hence, $d_2(u) \geq 1$ for each $u \in V(H_1)$. Let $V(H_1) = \{u_1, u_2, \ldots, u_{p_1}\}$. Then, $\kappa'(G) = |S| = \sum_{i=1}^{p_1} d_2(u_i) = \Big(\sum_{i=1}^{p_1-1} d_2(u_i)\Big) + d_2(u_{p_1}) \geq p_1 - 1 + d_2(u_n)$.

(3)

By (2) and (3), we conclude that $\kappa'(G) = \delta(G)$. □

For more properties about connectivities, the readers are encouraged to refer a textbook of Graph Theory, for example, [5].

7.4 Eulerian and Hamiltonian graphs

The first paper of Graph Theory was on the existence of an eulerian circuit.[6]

Definition 7.4.1. *An* eulerian circuit *of a graph is a circuit passing through all the edges of G.*

The idea of eulerian circuit was motivated by the well-known seven bridges problem. The aim of solving the problem can be modeled as finding a circuit passing through all the edges of the following graphs (multi-graph), Figure 7.4.1.

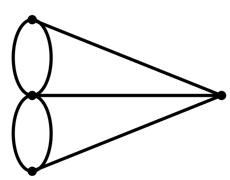


Figure 7.4.1. Seven Bridge Problem.

The paper published in 1941 provided an answer for this problem. Furthermore, the paper characterized the existence of such circuits, known as eulerian circuit now.

Theorem 7.4.1. *[6] A multi-graph G contains an* eulerian circuit *if and only if G is connected and each vertex is of even degree.*

Proof. The necessity of the assertion is easy to see, we prove the sufficiency by induction on the order of G. Clearly, it is true for graphs of small orders. Assume that the assertion is true for graphs of order less than n and let the graph we consider, G, be of order n.

Let $v \in V(G)$ and $deg(v) = 2s$. Also, let $N(v) = \{v_1, v_2, \ldots, v_{2s-1}, v_{2s}\}$. Now, we let $\widetilde{G}$ be obtained by deleting the vertex v and add s new edges $v_1v_2, v_3v_4, \ldots, v_{2s-1}v_{2s}$. Note that if there exist edges in G which are the same as there s new edges, we simply put them together as multi-edges. Hence, $\widetilde{G}$ is a graph whose vertices are of even degree. But $\widetilde{G}$ may have several component, say $\widetilde{G}_1, \widetilde{G}_2, \ldots, \widetilde{G}_t$. By induction, each of these components contains an eulerian circuit. Now, we may combining these circuits together by attaching them to the vertex v: Replace $v_{2i-1}v_{2i}$(new) with vv_{2i-1} and vv_{2i} for each $i = 1, 2, \ldots, s$. Thus, we have an eulerian circuit of G, starting from v and passing all the edges of each component $\widetilde{G}_i$, $i = 1, 2, \ldots, t$, except those new edges. □

The conditions for the existence of an eulerian circuit is quite simple. But, if we are given an eulerian graph i.e. a graph with eulerian circuit, finding the circuit is not that easy especially the graph is of large order and size. We do need a good algorithm to find the circuit, see [2] for example. It is worth of noting here that finding the number of different eulerian circuits in an eulerian graph is a challenging problem. The explicit formula for eulerian digraphs has been obtained, but for general multigraph the problem is still open. The readers may refer to [5] for the result of digraph version.

As an interesting note for eulerian graphs, the following song texted by *Donald A. Preece* provides the history of Graph Thoery.

1 Seven bridges spanned the River Pregel, many more than might have been expected; Knigsbergs wise leaders were delighted to have built such very splendid structures.

2 Crowds each evening surged twoards the river, people walked bemused across the bridges, pondering a simple-sounding challenge which defeated them and left them puzzled!

3 Here's the problem; see if you can solve it! Try it out at home on scraps of paper! Starting out and ending at the same spot, you must cross each bridge just once each evening.

4 All the folk in Knigsberg were frantic! All their efforts ended up in failure! Happily, a learn-ed mathematician had his house right there within the city.

5 Euler's mind was equal to the problem: "Ah", he said,"You are bound to be disheartened. Crossing each bridge only once per outing can't be done, I truly do assure you."

6 Laws of Nature never can be altered, we cant change them, even if we wish to. Nor can flooded rivers or great bridges Interfere with scientific progress.

7 War brought strife and ruin to the Pregel; bombs destroyed those seven splendid bridges. Eulers name and fame will, notwithstanding, be recalled with Knigsbergs for ever.

8 Thanks to Euler, Graph Theory is thriving. Year by year it flourishes and blossoms, fertilizing much of mathematics and so rich in all its applications.

9 Colleagues, let us fill up all our glasses! Colleagues, let us raise them now to toast the greatness and the everlasting glory of our Graph Theory, which we love dearly!

Indeed, Graph Theory plays a very important role in modern era, say complex network which gets involved in our daily life.

Another important notion of connectivity is the following.

Definition 7.4.2. *A cycle in a graph is called a hamiltonian cycle if this cycle passes through all vertices of the graph. A graph admits a hamiltonian cycle is known as a hamiltonial graph, or hamilton graph.*

Clearly, such a graph must be 2-connected since deleting any vertex can not disconnect the graph. Unlike the result obtained in characterizing the eulerian graphs, finding a hamiltonian cycle in a graph is not easy at all. So far, we are able to obtain quite a few sufficient conditions for a graph to contain a hamiltonian cycle, but none of these conditions are necessary. In fact, if algorithmic point of view is concerned, then finding a hamiltonian cycle in a 2-connected graph is an NP-hard problem.

Here, we present a sufficient condition for the existence of a hamiltonian cycle in a graph.

Theorem 7.4.2. *[12] Let G be a graph of order at least three. Then G contains a hamiltonian cycle if for any two non-adjacent vertices u and v, $deg_G(u) + deg_G(v) \geq |G|$.*

Proof. Let $|G| = p$. We prove the theorem by considering the existence of a minimal counter-example. That is, we assume that G is not a hamiltonian graph but $G+xy$ contains a hamiltonian cycle where x and y are not adjacent in G. This implies that G contains a path from x to y which contains all the vertices of G(hamiltonian path), see Figure 7.4.2. for an explanation. Since $deg_G(x) + deg_G(y) \geq p$, we conclude that there exists an $i \in \{2, \dots, p-2\}$ such that $v_0 \sim v_{i+1}$ and $v_{p-1} \sim v_i$. For otherwise $deg_G(x) + deg_G(y) \leq p-1$, a contradiction. But, with the adjacency mentioned above, we have a hamiltonian cycle $(x, v_1, v_2, \dots, v_i, y, v_{p-2}, \dots, v_{i+1})$. Again, a contradiction. □

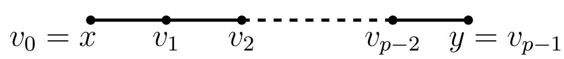


Figure 7.4.2. A hamiltonian path

A quick reflection, we notice that this condition is not necessary, for example, C_5, a cycle with 5 vertices. There are quite a few sufficient conditions in the literatures.[15] We end this section with a couple conditions which we list them as exercises. They are worth of trying to prove them.

Exercise 11. *Let G be a graph of order $p \geq 3$. Prove that if for every integer j with $1 \leq j \leq \frac{p}{2}$, the number of vertices of degree not exceeding j is less than j, then G is hamiltonian.*

Exercise 12. *Let $\beta(G)$ denote the maximum number of vertices which are mutually non-adjacent. Prove that if $\kappa(G) \geq \beta(G)$, then G is hamiltonian.*

Exercise 13. *Show that if G is hamiltonian, then for every nonempty set $S \subsetneqq V(G)$, $c(G-S) \leq |S|$. $\big(c(H)$ is the components of $H.\big)$*

Exercise 14. *Prove that if $||G|| \geq \frac{(p^2-3p+6)}{2}$ where $p = |G|$, then G is hamiltonian.*

7.5 Planar and non-planar graphs

Topological graph theory studies the "drawing" of a graph on a surface defined on $\mathbb{R}^3$. A proper drawing on a surface of a graph G with $|G| = p$ and $\parallel G \parallel = q$ follows the rules:

(1) There are p points on the surface which corresponds to the set of vertices in G; and

(2) There are q curves joining points defined above which correspond to the set of edges and they are pairwise disjoint except possibly for the endpoints.

The following are basic notions of a 2-manifold:

(1) A **2-manifold** is a connected topological space in which every point has a neighborhood homeomorphic to the open unit disk defined on $\mathbb{R}^2$.

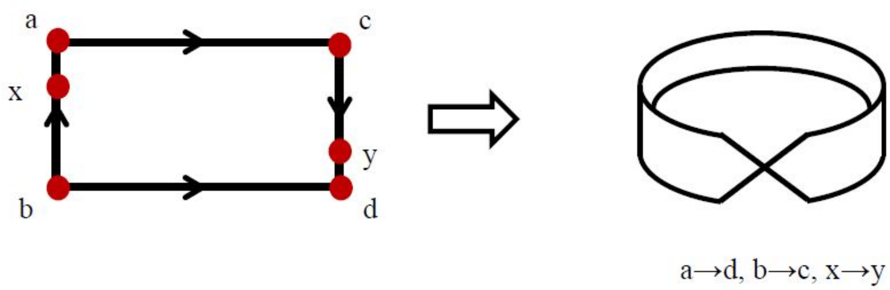
(2) A subspace M of $\mathbb{R}^3$ is **bounded** if there exists a positive real number K such that $M \subseteq \{(x, y, z) | x^2 + y^2 + z^2 = K\}$.

(3) Let $M \subseteq \mathbb{R}^3$ be 2-manifold. Then M is said to be **closed** if it is bounded and the boundary of M coincides with M.

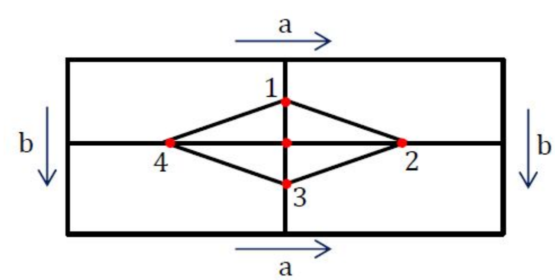
(4) Let $M(\subseteq \mathbb{R}^3)$ be a 2-manifold; M is said to be **orientable** if for every simple closed C on M, a clockwise sense of rotation is preserved by traveling once around C. Otherwise, M is non-orientable.

Definition 7.5.1 (**Orientable Surface**). *A surface is a compact orientable 2-manifold that may be thought of as a sphere on whih has been placed(inserted) a number of "handles"(holes). A sphere, denoted by S_0, is the surface of a 3-dimension ball. More precisely, $S_0 = \{(x, y, z)|x^2 + y^2 + z^2 = r^2, r \in \mathbb{R}^+\}$. S_1 is known as a torus, S_2 a double torus, and S_h is a surface obtained by adding h handles to S_0.*

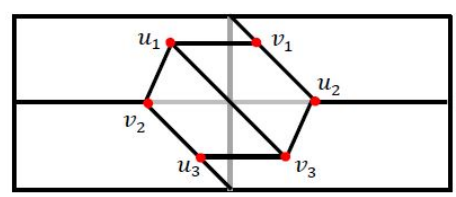
Definition 7.5.2 (**Non-Orientable Surface**). *A surface obtained by adding k cross-caps to S_0 is known as the non-orientable surface N_k. (A cross cap is obtained from Möbius band described in what follows.)*
Cross cap: Attach the boundary of a Möbius band to a cycle on S_0 to obtain a cross cap.



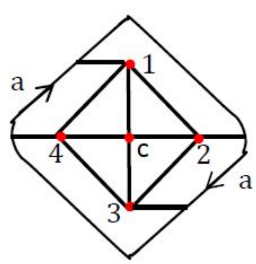
Definition 7.5.3 (**Embedding or Imbedding, 2-cell embedding**). *An embedding of a graph in a surface is a continuous $1-1$ function from a topological representation of the graph into the* ***surface****. If every region of the embedding is homeomorphic to a 2-dim open disc, then the embedding is a* ***2-cell embedding****.*



Embedding of K_5 on S_1

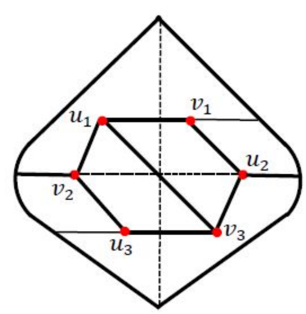


Embedding of $K_{3,3}$ on S_1



Embedding of K_5 on N_1

(N_1 is also known as a projective plane.)



Embedding of $K_{3,3}$ on N_1

Definition 7.5.4. *If a graph G can be embedded in S_0, then G is a planar graph.*

Theorem 7.5.1 (**Euler's Formula**). *Let G be a* ***connected planar*** *graph which has p vertices, q edges and r regions. Then $p - q + r = 2$.*

Proof. By induction on q. Since G is connected, $q \geq p - 1$. If $q = p - 1$, then G is a tree. Hence, $r = 1$. and the proof follows.
Now assume that the assertion is a true for $q \geq p$ and let $\parallel G \parallel = q + 1$ and G has r regions. Let e be a cycle edge of G. Then $G - e$ is connected which has $r - 1$ regions, p vertices and $q - 1$ edges.
By induction, $p - (q + 1) + r = p - q + (r - 1) = 2$. This concludes the proof. $\square$

Corollary 7.5.2. *Let G be a planar graph which has k components, p vertices, q edges and r regions. Then $p - q + r = 1 + k$.*

Proof. By induction on k. k = 1 is true by Theorem 7.5.1. Assume the assertion is true for k. Let G be a graph with p vertices, q edges and r regions, and G has $k + 1(\geq 2)$ components. Now let $\widetilde{G}$ has p vertices, $q + 1$ edges, r regions and k components.(Connect two components with one extra edge.) Hence, $p - (q + 1) + r = 1 + k$. This implies $p - q + r = 1 + (k + 1)$. The proof follows. $\square$

Definition 7.5.5 (**Maximal planar graph**). *A planar graph is maximal if $\forall\, u, v \in V(G)$, $uv \notin E(G)$, $G + uv$ is not planar.*

Theorem 7.5.3. *If G is a maximal planar (p, q)-graph, then $q = 3p - 6$.*

Proof. If G is maximal, then $3r = 2q$ since each region is a triangle. $\square$

Corollary 7.5.4. *If G is a planar graph, then $q \leq 3p - 6$.*

Theorem 7.5.5. *If G is a planar graph with girth g, then G has at most $\frac{g}{g-2}(p - 2)$ edges, i.e., $q \leq \frac{g}{g-2}(p - 2)$.*

Proof. $gr \leq 2q$; $p - q + r = 2$; $p - q + \frac{2q}{g} \geq 2$; $p - 2 \geq q(1 - \frac{2}{g})$; $p - 2 \geq q\frac{g-2}{g}$.
We can use the above properties to show the existence of platonic solids: regular planar graphs. $\square$

Theorem 7.5.6. *There are exactly five regular polyhedra.*

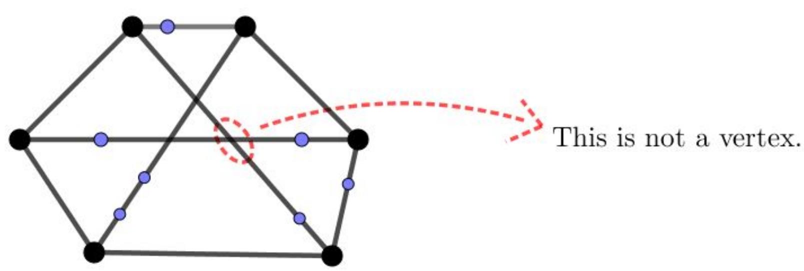
Proof. Clearly, a platonic solid is a regular planar graph. Let it be G.
Then G : k-regular, p vertices, q edges and r faces (have length l).
$kp = 2q = rl$, $p = \frac{2}{k}q$, $r = \frac{2}{l}q$.

$p - q + r = 2$
$\Rightarrow \frac{2}{k}q - q + \frac{2}{l}q = 2$
$\Rightarrow q(\frac{2}{k} - 1 + \frac{2}{l}) = 2,\ \frac{2}{k} - 1 + \frac{2}{l} > 0$
$2l - kl + 2k > 0 \Rightarrow kl - 2k - 2l < 0$
$(k-2)(l-2) = kl - 2l - 2k + 4 < 4.$
Now, $k \leq 5$ and thus $l \leq 5(?)$ (Dual graph is also planar!).
By direct checking the graph exists if and only if
$(k, l) \in \{(3,3), (3,4,), (3,5), (4,3), (5,3)\}$. This concludes the proof. □

(k, l)	q	p	r	Name
(3,3)	6	4	4	Tetrahedron
(3,4)	12	8	6	Cube
(4,3)	12	6	8	Octahedron
(3,5)	30	20	12	Dodecahedron
(5,3)	30	12	20	Icosahedron

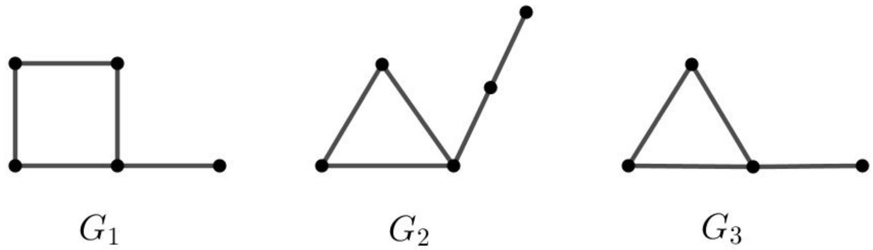
We may give a characterization of planar graphs. First, we need some definitions.

Definition 7.5.6. *A subdivision of a graph is obtained from it by replacing edges with pairwise internally disjoint paths.*



Definition 7.5.7. *A graph H is said to be homeomorphic from G if either $H \cong G$ or H is isomorphic to a subdivision of G. A graph G_1 is homeomorphic with G_2 if there exists a graph G_3 such that G_1 and G_2 are both homeomorphic from G_3.*

Both of G_1 and G_2 are homeomorphic from G_3.



Proposition 7.5.7. *If a graph G has a subgraph that is homeomorphic from K_5 or $K_{3,3}$, then G is nonplanar.*

Proof. If follows by the fact that both K_5 or $K_{3,3}$ are not planar graphs. □

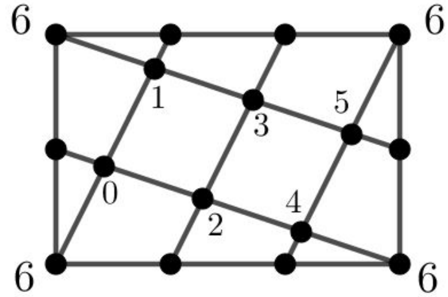
The following theorem is well-known. We shall omit its proof here. The interested readers may refer to [15] for more details.

Theorem 7.5.8 (Kuratowski [1930])**.** *A graph is planar if and only if it does not contain a subgraph which is homeomorphic from K_5 or $K_{3,3}$.*

The graphs which are not planar is called non-planar. In general, we use genus to describe how they can be drawn on a surface.

Definition 7.5.8. *The orientable genus of a graph G, $\gamma(G)$, is the minimum genus of a surface in which G cab be embedded. The non-orientable genus of a graph G, $\overline{\gamma}(G)$, is the minimum non-orientable genus of a surface (crosscaps) in which G can be embedded.*

e.g. $\gamma(K_5) = \gamma(K_6) = \gamma(K_7) = 1.$
$\overline{\gamma}(K_5) = 1 = \overline{\gamma}(K_{3,3}).$



Theorem 7.5.9 (Euler-Poincaré (for pseudo-graphs))**.** *Let G be a (p, q) graph which has a 2-cell embedding in an orientable surface of genus n. Then $p - q + r = 2 - 2n$ where r is the number of regions.*

Proof. By induction on n and it is true for $n = 0$ which is a direct consequence of Euler's formula. Now, assume the assertion is true for the surface with genus less than n and G is a graph which has an orientable 2-cell embedding on S_n.

First, draw a cycle C along a handle which does not meet any vertex of $V(G)$, see figure below. Let C intersect m edges of G, see the following figures.

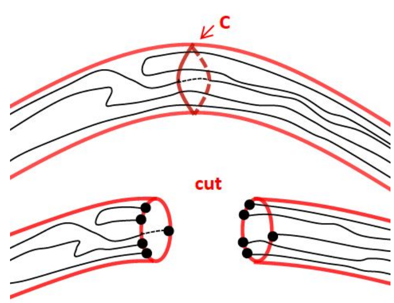


Figure 7.5.3

Moreover, let the total number of "intersections" be k. Now, by cutting off the handle from C and paste two regions to the planes they form from cut we obtain a surface with one less handles. Also, let the points of intersection be new vertices, then the new graph has $p' = p+2k$ vertices, $q' = q+3k$ edges and $r' = r + k + 2$ regions. Since the new graph is embedded on a surface of genus $n - 1$, $p' - q' + r' = 2 - 2(n - 1)$ and thus $p - q + r = 2 - 2n$. □

Observe that the above theorem does extend the Euler's formula when the graph is planar, in that case, $n = 0$. Moreover, we also have a version for non-orientable. For the limitation of our content, we omit its proof.

Theorem 7.5.10. *Let the non-orientable genus of G be h. Then $p - q + r = 2 - h$.*

Topological graph theory is very fruitful and also plays an important role in studying the structure of graphs. For more information and theoretical arguments, please refer to [5].

Exercise 15. *Prove that the Petersen graph is not a planar graph.*

Exercise 16. *If we draw a non-planar graph on a plane, then there are crossings of edges. The minimum of crossings of a drawing on a plane of a graph G is denoted by $cr(G)$. Show that $cr(K_6) = 3$.*

Exercise 17. *Let G be a planar graph. Prove that G can be drawn on a plane such that each edge is a straight line segment.*

7.6 Graph Colorings

1. Map Coloring Problem

It was happened on Oct. 23, 1852, Augustus DeMorgan sent a letter to William Rowan Hamilton in Trinity College at Dublin. The letter mentioned that a student of DeMorgan guessed 4 colors are enough to color the compartments of England differently so that figures with any portion of common boundary are colored with different colors.

The student's name is Frederick Guthrie and he learned this idea from his brother Francis Guthrie who actually provided an incomplete proof of the above fact. Nowaday, this is the so-called Map Coloring problem. People believe that this problem is indeed the most influential one in the development of modern Graph Theory. As a matter of fact, because of the proof obtained by Appel and Haken[1] did use the assistance of computers, it also accelerates the development of Computer Sciences.

In this section, we shall introduces several types of graph colorings. Each of them does find its applications in Science and Engineers.

Definition 7.6.1 (**Region Coloring**). *A plane graph G is said to be k-region colorable if the regions of G can be colored with k or fewer colors so that adjacent regions are colored differently.*

The problem mentioned by DeMorgan was formulated as the four color conjecture: Every map (plane graph) is 4-region colorable. Now, it is known as the 4-color theorem, 4CT in short.

To study the map coloring, the following idea plays an important role.

Definition 7.6.2 (**Dual Graph**). *Let G be a plane graph with r regions. Then, a dual graph G^* of G is defined as follows:*

(1) $V(G^*) = \{v_1, v_2, \ldots, v_r\}$ *where each vertex represents a region,*

(2) two vertices are adjacent if they share a common boundary.

For example, let G be a graph in Figure7.6.1(a), then G^* is obtained in Figure7.6.1(b). $|G^*| = r$ and $\| G^* \| = \| G \|$.

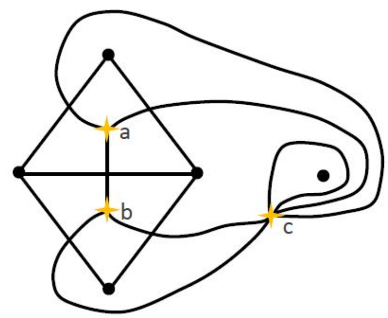


Figure 7.6.1(a), G

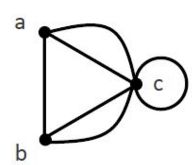


Figure 7.6.1(b), G^*

It is not difficult to see that the map coloring can be transferred to find a coloring on its dual graph. Since multiple edges and loops do not affect the coloring of vertices, we shall delete all of them and consider the **underlying graph** of G^*. The underlying graph of Figure 7.6.1(b) is simply a triangle. So, as long as we have a proper coloring of the underlying graph on its vertices, we have a solution for map coloring.

Definition 7.6.3 (Vertex Coloring)**.** *A vertex k-coloring of G is a mapping φ: $V(G) \to \{1, 2, \ldots, k\}$ such that $\varphi(u) \neq \varphi(v)$ if uv is an edge of G. The minimum integer k such that G has a vertex k-coloring is called the chromatic number of G, denoted by $\chi(G)$.*

For convenience, we shall use "k-coloring" if G has a k-coloring.

The following facts are easy to see.

(F1) If $G \geq K_k$, then $\chi(G) \geq k$. (K_k is a clique in G.)

(F2) If G is a nontrivial bipartite graph, then $\chi(G) = 2$. Hence, if G is a tree, then $\chi(G) = 2$.

(F3) If G is vertex-colored, then two vertices are not adjacent provided they receive the same color.

(F4) The set of vertices with the same color in a colored graph is called a color class and each color class is an **independent set**, i.e., a set of mutually non-adjacent vertices.

(F5) $\chi(K_n) = n$ and $\chi(C_n) = 2$ if and only if n is even. If n is odd, then $\chi(C_n) = 3$.

(F6) If a graph is k-colorable, then it is h-colorable for each $h \geq k$. Moreover, its subgraph is also h-colorable.

(F7) If $\Delta(G)$ is the maximum degree of G, then $\chi(G) \leq \Delta(G) + 1$.

The above fact can be proved by using the so-called greedy coloring. Mainly, we color the vertices by following the arithmetic order of $1, 2, \ldots, \Delta(G)+1$, and we color the next vertex by using the smallest integer we can use if it is available.

The following theorem is well-known. Interested readers may refer to [15] for its proof.

Theorem 7.6.4 (Brooks' Theorem,[3])**.** *For any graph G, $\chi(G) \leq \Delta(G)+1$ and $\chi(G) \leq \Delta(G)$ provided that G is not a complete graph or an odd cycle.*

To determine $\chi(G)$ of a graph G is in general very difficult. There are quite a few good upper bounds which are known. The following result can be applied without too much difficulty.

First, we need a definition.

Definition 7.6.5. (Degenerate) *A graph G is called k-degenerate if each subgraph of G, H, $\delta(H) \leq k$.*

Example. *A planar graph is 5-degenerate and an outer planar graph is 2-degenerate.*

Theorem 7.6.6. *If G is k-degenerate, then $\chi(G) \leq k + 1$.*

Proof. By induction on the order of G. Clearly, it is true for small graphs. Assume that G is k-degenerate and G is of order n. Then, G has a vertex v of order $\delta(G)$. Now, consider $G - v$. By induction, since $G - v$ is a subgraph of G and $G - v$ is also k-degenerate, $\chi(G - v) \leq k + 1$. Let φ' be a $(k + 1)$-coloring of $G - v$. Then, by coloring v the color which are missing in the neighborhood of v in G (at most k of them), we obtain a $(k+1)$-coloring of G. □

So, it takes no time to conclude that a planar graph is 6-colorable. But, some effort is needed to use 5 colors.

Theorem 7.6.7. *Let G be a planar graph, then $\chi(G) \le 5$.*

Proof. By induction on $|G|$. Assume that G is a planar graph of order n and the assertion is true for all graphs of smaller orders. First, if $\delta(G) = 3$ or 4, let $deg(v) = \delta(G)$. Since $\chi(G - v) \le 5$, we may color v by using the colors missing in its neighbors. Hence, the case left is the case $\delta(G) = 5$. Without loss of generality, we may let v be adjacent to v_1, v_2, v_3, v_4 and v_5 as in Figure 7.6.2. Moreover, they are colored with five different colors. For otherwise, we may color v with the color missing. Let $\varphi(v_i) = i$, $i = 1, 2, 3, 4, 5$ in the coloring of $G - v$. Now, we claim that we can recolor v_i to "3" or recolor v_2 to "4".

Starting from v_1, consider the occurrence of color 3 in its neighbors. If none of them are colored 3, then v_1 can be recolored with 3. Otherwise, we have a neighbor of v_1, v_2, whose color is 3. Then consider the neighbor of u_2 whose color is 1(except v_1). If 1 does not occur, then, recolor u_2 by 1 and v_1 by 3. On the other hand, 1 does occur in u_3. Next, we search 3 in the neighbors of u_3(except u_2). Hence, the situation that we are not able to recolor them is the existence of a chain: $v_1 - v_2 - v_3 - \ldots - v_3$ and the colors 1 and 3 occur alternatively. If this happens, then we consider the chain of colors 2 and 4 starting from v_2. Due to the planarity of G, the chain will not end at v_4. Therefore, we can recolor the chain by using $4, 2$ alternatively. Following the consequence, color 2 is missing in the neighbors of v and the proof follows by assigning 2 to v. □

Notice that the above idea was known as Kempe-chain method. It was applied to prove the 4 color theorem in the late 19 century[10]. Unfortunately, there is a flaw in the proof. The correct proof takes another 80 years to finish with the assistance of computers.[1] Later, some improvement was made but computer-aid is still there[14]. We do wish that a proof without using computer checking can by obtained in the near future.

Exercise 18. *Find a graph G which contains no triangles C_3 and $\chi(G) = 4$.*

Exercise 19. *Let $m(G)$ denote the length of a longest path in a graph G. Prove that $\chi(G) \le 1 + M(G)$.*

Exercise 20. *A graph is called critically n-chromatic if* $\chi(G) = m$ *but for each* $v \in V(G)$, $\chi(G - v) = m - 1$.

Exercise 21. *A coloring* φ *of a graph* G *is called an equitable coloring if for any two colors* i *and* j *used in the coloring,* $\Big||\varphi^{-1}(i)| - |\varphi^{-1}(j)|\Big| \leq 1$. *Find three examples(classes of graphs) that* G *has a* 2*-coloring but not an equitable* 2*-coloring.*

2. Edge-coloring

An equally important type of coloring is the edge-coloring of a graph.

Definition 7.6.8 (Edge-coloring)**.** *A* k*-edge-coloring of a graph* G *is a mapping* $\pi\colon\ E(G) \to \{1, 2, \ldots, k\}$ *such that* $\pi(e) \neq \pi(f)$ *provided* $e, f \in E(G)$ *and* $e \cap f \neq \phi$. *The minimum integer* k *such that* G *has a* k*-edge-coloring is called the chromatic index of* G, *denoted by* $\chi'(G)$.

The following facts are easy to check.

(F1) $\chi'(G) \geq \Delta(G)$.

(F2) Each color of an edge-coloring induces a matching.

(F3) If n is odd, then $\chi'(C_n) = 3$ and $\chi'(K_n) > n - 1$.

The reason of $\chi(K_n) > n-1$ when n is odd is the following. For example, in K_5, each color class(edges of the same color) can have at most two edges, so we need at least 5 colors to color the edges of K_5. Hence, 4 colors are not enough.

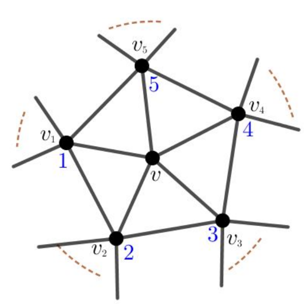


Figure 7.6.2 Degree 5 case

Definition 7.6.9 (Over-full graphs)**.** *A graph G is said to be over-full if* $\lfloor \frac{|G|}{2} \rfloor \cdot \Delta(G) < \parallel G \parallel$.

Clearly, K_5 is an over-full graph so are the complete graphs and cycles of odd order. The following result is a direct consequence of the definition.

(F4) If G is over-full, then $\chi'(G) \geq \Delta(G) + 1$.

One more important observation is about the multigraphs. In vertex coloring, multi-edges do not affect the number of colors, but in edge coloring, they are significant. In order to handle this case, we need to define the multiplicity of a graph.

Definition 7.6.10 (Multiplicity)**.** *The multiplicity of a multigraph G is the maximum number of multi-edges between two vertices of G, denoted by* $\mu(G)$.

For example, the multiplicity of the graph in Figure 7.6.3 is 2.

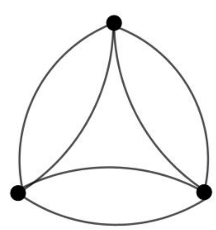


Figure 7.6.3 A multiplicity 2 graph $2K_3$

So it is not difficult to see that the above graph $2K_3$ has chromatic index 6. To determine the upper bound of $\chi'(G)$, Vizing[16] proved the following break through theorem. We shall omit the proof here. Interested readers may refer to[5] for the details.

Theorem 7.6.11. [16] *If G is a multigraph, then* $\chi'(G) \leq \Delta(G) + \mu(G)$.

Theorem 7.6.12. *If G is a simple graph, then* $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

We shall focus on simple graphs in what follows. For characterization of graphs corresponding to their chromatic indices, we have the classes of graphs.

Definition 7.6.13. *A graph G is called Class 1 if $\chi'(G) = \Delta(G)$ and Class 2 otherwise.*

Therefore, a complete graph of order n is of Class 1 if and only if n is even.(Exercise) But, for a bipartite graph, König[11] proved the following result. We shall use the idea of SDR to prove the theorem.

Theorem 7.6.14. *A bipartite graph is of Class 1.*

Proof. Let G be a graph with $\Delta(G) = k$. Then, there exists a k-regular graph $\widetilde{G}$ which contains G as a subgraph(Exercise). It suffices to prove that $\chi'(\widetilde{G}) = k$ and equivalently $E(\widetilde{G})$ can be partitioned into k perfect matchings (a spanning matching of $\widetilde{G}$).

Now, let $\widetilde{G} = (A, B)$. Clearly, $|A| = |B|$. Let $A = \{a_1, a_2, \ldots, a_n\}$ and $S_i = N_G(a_i)$, $i = 1, 2, \ldots, n$. We claim that $\mathbb{S} = \{S_1, S_2, \ldots, S_n\}$ does have an SDR. Note that an SDR is corresponding to a perfect matching of $\widetilde{G}$. Let $\{S_{i_1}, S_{i_2}, \ldots, S_{i'_k}\}$ be an arbitrary collection of sets in $\mathbb{S}$, $1 \leq k' \leq n$. Consider $\bigcup_{j=1}^{k'} S_{i_j}$. The occurrence of vertices in B is therefore in total $k \cdot k'$. Notice that some of them may occur several times. Since each of them can occur at most k times (k regular), this implies that $\left|\bigcup_{j=1}^{k'} S_{i_j}\right| \geq \frac{k \cdot k'}{k} = k'$. Hence $\mathbb{S}$ has an SDR. Now, we delete the perfect matching and continue the above process to obtain k perfect matchings. This concludes the proof. □

Theorem 7.6.14 is known as the König's Theorem today. It was known as the "Marriage Problem".[11] As a matter of fact, many applications in real world can be modeled as an edge-coloring problem, such as scheduling problems. Not only applications, theoretically the edge-coloring plays an important role in learning the graph structure. Here, we mention one more important result.

Theorem 7.6.15. *If G is a 3-regular connected planar graph, then $\chi'(G) = 3$.*

Proof. Since G is planar, the map coloring of G uses at most 4 colors. Let the set of 4 colors be $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Now, we define an edge-coloring π of G by letting $\pi(e) = (i_1, j_1) \oplus (i_2, j_2)$ where $\oplus$ is an operation in $\mathbb{Z}_2 \times \mathbb{Z}_2$ and e is the boundary between two regions colored (i_1, j_1) and

(i_2, j_2) respectively. Since the vertex coloring is proper, i.e., $(i_1, j_1) \neq (i_2, j_2)$, we only use at most 3 colors for the edge coloring π. Now, we claim π is also a proper coloring. Let $e \cup f \neq \phi$, i.e., e and f are incident edges. Then, e and f are boundaries of one common region R_1, and two distinct regions R_2 and R_3, see Figure7.6.4. Hence $\pi(e) \neq \pi(f)$.

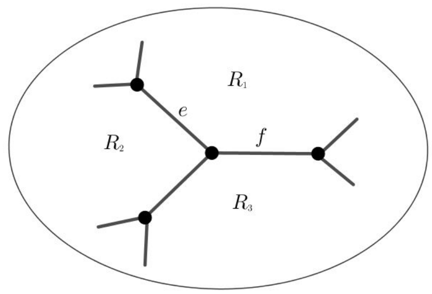


Figure 7.6.4 3-regular planar graph

□

Without using 4-color theorem, the proof of the above theorem remains a very challenging one. Indeed, we have much more to say about the edge-coloring of a graph. But, for the content of this chapter, we stop here.

Exercise 22. *Prove that $\chi'(K_n) = n - 1$ if n is even and $\chi'(K_n) = n$ if n is odd.*

Exercise 23. *Prove that Pertersen graph is of Class 2.*

Exercise 24. *Prove that if G is a bipartite graph, then there exists a supergraph $\widetilde{G}$ of G which is bipartite and $\Delta(G)$-regular. (If G is a subgraph of $\widetilde{G}$, then $\widetilde{G}$ is called a supergraph of G.)*

Exercise 25. *Prove that for each even integer $n \geq 10$, an $(n-3)$-regular graph of order n is of Class 1.*

Exercise 26. *Let K_{n_1,n_2,n_3} be the complete 3-partite graph. Determine the conditions on n_1, n_2 and n_3 such that K_{n_1,n_2,n_3} is of Class 2.*

Exercise 27. *Prove that if G has a k-edge-coloring, then G also has an equalized k-edge-coloring, i.e., the difference between the sizes of color classes is at most 1.*

There are quite a few different types of colorings for graphs. One of them is called total coloring. We can color the vertices and edges simultaneously. For vertices, it is a vertex coloring and for edges, it is an edge coloring and also incident elements receive distinct colors. We may also color the regions and the boundaries of an embedded graph on some surfaces such that the color of a region is different from its adjacent regions and also different from its boundaries. The interesting readers may refer to [7] for a detailed survey.

7.7 Random Graphs

The theory of random graphs was founded by Erdös and Rényi around 1960 after the discovery of probabilistic methods used in tackling extremal problems in graph theory by Erdös earlier. Nowadays, it becomes one of the most important tools in the study of graph structures. In general, there are three models:

Model A **:** $G(n, p)$ or $G(n, P = p)$, $0 \leq p \leq 1$.
The probability of the existence of an edge (independently) is p and the graph induced by using existent edges is G_p.

Model B **:** $G(n, M)$
We assume that the probability of an M edges graph G_M is equal. Therefore, an M edges graph H occurs with probability $\frac{1}{\binom{N}{M}}$, denoted by $P(G_M = H) = \binom{N}{M}$ where $N = \binom{n}{2}$.

Model C **:** $\widetilde{G} = (e_1, e_2, \ldots, e_N)$ where $e_t \in E(G_t) \backslash E(G_{t-1})$ and G_t is obtained in Model B. Hence, we have a sequence of random graphs $G_0 \leq G_1 \leq \ldots G_N$.

1. Basic Notions

Definition 7.7.1 (Discrete Probabilistic Space, D.P.S.)**.** *A D.P.S. is an ordered pair* (S, f) *where* S *is a countable set and* $f : S \to \mathbb{R}$ *satisfying*
(i) $0 \leq f(x) \leq 1$, *and* *(ii)* $\sum_{x \in S} f(x) = 1$.

Definition 7.7.2 (The probability of an event $A \subseteq S$)**.** *Let* (S, f) *be a D.P.S.. Then the probability of* $A \subseteq S$ *is* $P(A) = \sum_{x \in A} f(x)$.

Definition 7.7.3 (Independent events)**.** *If $P(A \cap B) = P(A)P(B)$, then A and B are independent events.*

Definition 7.7.4 (Random variables)**.** *Let (S, f) be a D.P.S.. Then $\mathbb{X} :\to \mathbb{R}$ is a random variable where we use $\mathbb{X} = k$ to denote an event with image k and $K = \{x \in S \,|\, \mathbb{X}(x) = k\}$.*
e.g. Let $S = [1, 6]^2$ and $f(x, y) = \frac{1}{36}$ for each $(x, y) \in [1, 6]^2$
$\mathbb{X}((x, y)) = x+y$, $k = 7 \Rightarrow (\mathbb{X} = 7) = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$.

Definition 7.7.5 (Expectation)**.** *Let $\mathbb{X}$ be a random variable. Then the expectation of $\mathbb{X}$, $\mathbb{E}(\mathbb{X}) = \sum\limits_k k \cdot p(\mathbb{X} = k)$. (We define $P(\mathbb{X} = h) = 0$ if h is not in the image of $\mathbb{X} : S \to \mathbb{R}$.)*
e.g. (Continued), $\mathbb{X} = 7$.

$$\begin{aligned}
\mathbb{E}(\mathbb{X}) &= 2 \cdot \frac{1}{36} + 3 \cdot \frac{1}{18} + 4 \cdot \frac{1}{12} + 5 \cdot \frac{1}{9} + 6 \cdot \frac{5}{36} + 7 \cdot \frac{1}{6} \\
&\quad + 12 \cdot \frac{1}{36} + 11 \cdot \frac{1}{18} + 10 \cdot \frac{1}{12} + 9 \cdot \frac{1}{9} + 8 \cdot \frac{5}{36} \\
&= 14 \cdot (\frac{1}{36} + \frac{1}{18} + \frac{1}{12} + \frac{1}{9} + \frac{5}{36} + \frac{1}{12}) \\
&= 14 \cdot \frac{1 + 2 + 3 + 4 + 5 + 3}{36} \\
&= 7.
\end{aligned}$$

Lemma 7.7.6 (Pigeon-hole principle of Expectation)**.** *Let $\mathbb{X}$ be a random variable of a D.P.S. Then, there exists a $y \in S$ such that $\mathbb{X}(y) \geq \mathbb{E}(\mathbb{X})$.*

Lemma 7.7.7 (Linear Property of Expectation)**.** *Let $X, X_1, X_2, \ldots, X_m$ be random variables such that $X = \sum\limits_{i=1}^{m} X_i$. Then, $\mathbb{E}(X) = \sum\limits_{i=1}^{m} \mathbb{E}(X_i)$.*

Definition 7.7.8 (Indicator Random Variable)**.** *An indicator random variable is a random variable X such that $\mathbb{X} : S \to \{0, 1\}$ (instead if $\mathbb{R}$).*
(Note): A random variable $\mathbb{X}$ can be written as a sum of $|G|$ indicator random variables

$$X_v = \begin{cases} 1 & if\, v \in \mathbb{X}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 7.7.9 (Szele, 1943)**.** *There exists a tournament T_n such that T_n has at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.*

Proof. There are $n!$ possible Hamiltonian (undirected) paths and the probability of an undirected Hamiltonian path (H. path) is a directed H. path is $\frac{1}{2^{n-1}}$. Therefore $\mathbb{E}(X) = n! \cdot \frac{1}{2^{n-1}}$. This concludes the proof. □

(Note): How many are they? In 1990, Alon proved that the number of H. paths is at most $\frac{n!}{2-o(1)}$.

Theorem 7.7.10. $\alpha(G) \geq \sum\limits_{v \in V(G)} \frac{1}{1+deg_G(v)}$.

Proof. Use $1, 2, \ldots, |G|$ to label the vertices of the set $V(G)$ randomly call it φ. Let $v_0 \in S$ (independent set) if $\varphi(v_0) = \min\{\varphi(x) | x \in N[v_o]\}$. So, the probability is $\frac{1}{1+deg_G(v_0)}$ and the expectation value is $\sum\limits_{v \in V(G)} \frac{1}{1+deg_G(v)}$. □

Definition 7.7.11 (Dominating Set)**.** *Let $S \subseteq V(G)$. Then, S is said to be a dominating set of G if for all $v \in V(G)$, v is adjacent to a vertex in S.* $\min\{|S| \Big| S$ *S is a dominating set of G*$\} = D(G)$ *is the domination number of G.*

Theorem 7.7.12 (Alon, 1990)**.** *Let $|G| = n$. Then $D(G) \leq \frac{n(l+ln(\delta(G))+1)}{\delta(G)+1}$.*

Proof. Let S be a subset of $V(G)$ with the probability of each vertex $p =_{def} \frac{ln(\delta(G)+1)}{\delta(G)+1}$. Let $T = \{x \Big| x \notin S,\ N(x) \cap S = \phi\}$. Then $S \cap T$ is a dominating set of G. By assumption, $E(|S|) = n \cdot p$ and $E(|T|) \leq n \cdot (1-p)^{\delta(G)+1}$. Note here that $|S|$ and $|T|$ are random variables. Since $(1-p)^{\delta(G)+1} \leq e^{-p(\delta(G)+1)}$, $E(|S|+|T|) \leq np+ne^{-p(\delta(G)+1)} = n(p+\frac{1}{\delta(G)+1}) = n(\frac{1+ln(\delta(G)+1)}{\delta(G)+1})$. This implies that there exists a dominating set with at most $n(\frac{1+ln(\delta(G)+1)}{\delta(G)+1})$ vertices. □

2. Almost all graphs

We use G^n to denote the distribution of graphs of order n. Let q_n be the probability of the existence os "Property Q".

Definition 7.7.13. *If $\lim\limits_{n\to\infty} q_n = 1$, then we say "$Q$" almost always holds or in this case, almost all graphs have property "Q".*

In what follows, let $\mathbb{X}$ be an Integer-Valued Random Variable.

Lemma 7.7.14 (Markov's Inequality)**.** *Let $p_k = P(\mathbb{X} = k)$, $k \geq 0$. Then, $P(\mathbb{X} \geq t) \leq \frac{\mathbb{E}(X)}{t}$. Moreover, if $\mathbb{E}(X) \to 0$, then $P(X = 0) \to 1$.*

Proof. $E(X) = \sum_{k\geq 0} k \cdot p_k \geq \sum_{k\geq t} k \cdot p_k \geq t \cdot \sum_{k\geq t} p_k = tP(\mathbb{X} \geq t)$. □

The following result is very interesting.

Theorem 7.7.15. *Let $0 < p \leq 1$. Then almost all graphs are of diameter 2.*

Proof. Let $\mathbb{X}_{i,j}$ be the indicator random variable such that

$$\mathbb{X}_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ do not have a common neighbor; and} \\ 0 & \textit{otherwise.} \end{cases}$$

Note that the probability of "v_i and v_j do not have a common neighbor" is equal to $(1-p^2)^{n-2}$, hence $P(\mathbb{X}_{i,j} = 1) = (1-p^2)^{n-2}$. By def. $\mathbb{X} = \sum_{i\neq j} \mathbb{E}(\mathbb{X}_{i,j}) = \binom{n}{2} \cdot (1-p^2)^{n-2}$. Since $\lim_{n\to\infty} \binom{n}{2}(1-p^2)^{n-2} = 0$, $\mathbb{E}(\mathbb{X}) \to 0$. This implies that $P(\mathbb{X} = 0) \to 1$, i.e., almost every pair of distinct vertices v_i and v_j have a common neighbor. This concludes the proof. □

Remark.

(1) Here, p is a constant!

(2) If G^p has n vertices, then G^p has $p \cdot \binom{n}{2}$ edges.

(3) p may be different as n changes; this is the idea of $p(n)$ which will be introduced next.

Definition 7.7.16 (Monotonic property)**.** *Let Q be a property. If G has property Q and for each edge $e \in E(\overline{G})$, $\overline{G}$ also has the property Q, then G is a monotonic property.*
e.g. If Q is the property "diameter 2", then Q is a monotonic property. Clearly, planarity is not a monotonic property.

Definition 7.7.17. *Let Q be a monotonic property and $t(n)$ be a function of n such that*

(i) $\frac{p(n)}{t(n)} \to 0 \Rightarrow$ Almost all graphs in G^p has no property Q, and

(ii) $\frac{p(n)}{t(n)} \to \infty \Rightarrow$ Almost all graphs G^p has property Q.

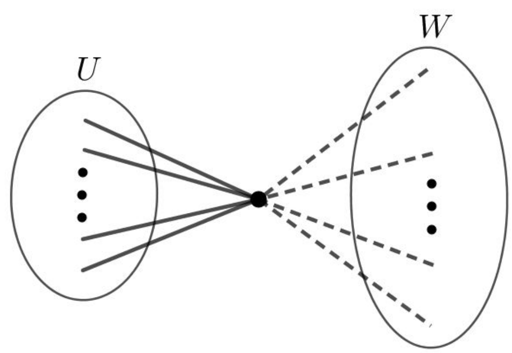
Then $t(n)$ is called a threshold probability function of Q.

Theorem 7.7.18. *For every constant $p \in (0,1)$ and every graph H, almost all graphs G^p contains an induced copy of H.*

Proof. Let H be given and $|H| = k$. Let U be a set of k (fixed) vertices of G then $< U >_G \cong H$ with a certain probability $r > 0$. (r depends on p, not n?) Now, G contains a collection of $\lfloor \frac{n}{k} \rfloor$ disjoint sets U_i of size "k". So,the probability that none of $< U_i >_G$ is isomorphic to H is $(1 - r^{\lfloor \frac{n}{k} \rfloor})$ which is going to "0" as $n \to \infty$. □

Theorem 7.7.19. *For every constant $p \in (0,1)$ and $i, j \in \mathbb{N}$, almost all graphs G^p has the property $P_{i,j}$ where $P_{i,j}$ is the property that for any disjoint vertex sets U and W with $|U|$ and $|W|$ with $|U| \leq i$ and $|W| \leq j$, a vertex $v \notin U \cup W$ that is adjacent to all the vertices of U but to none of vertices in W.*

Proof. The probability that $v \in V(G) \backslash (U \cup W)$ is adjacent to U but not to W is $p^{|U|} q^{|W|} \geq p^i q^j$. $|U| \leq i$, $|W| \leq j$.



Hence, the probability that no suitable v exists for these U and W is $(1 - p^{|U|q^{|W|}})^{n-|U|-|W|} \leq (1 - p^i q^j)^{n-i-j}$ where $(1 - p^i q^j)$ enlarge, and $n - i - j$ diminish, for $n \geq i + j$. Since the number of $< U, W >$ pairs is at most n^{i+j}, the total probability is $n^{i+j}(1 - p^i q^j)^{n-i-j} \to 0$ (i, j are constants!) as $n \to \infty$. □

So far, we consider the random graphs with fixed probability p for each edge. But, the probability p may be different when the number of vertices n is getting larger. Hence, instead of a fixed p, we have $p(n)$ to denote the probability of an edge. The following table can be found in [16].

$p(n)$	Graphs G
n^{-2}	No edges
$n^{-\frac{3}{2}}$	G has a tree component.
n^{-1}	G contains a cycle.
$\frac{log\, n}{n}$	G is connected.
$\frac{(1+\epsilon)log\, n}{n}$	G has a hamiltonian cycle.

Figure 7.7.1 Threshold functions p(n)

Exercise 28. *Let G be a graph of order n. Find a deterministic algorithm to prove that the domination number of G, $D(G)$ is at bounded above by* $\frac{n(1+ln(\delta(G)+1))}{\delta(G)+1}$.

Exercise 29. *For every constant $0 < p < 1$ and $k \in \mathbb{N}$, prove that almost all graphs are k-connected.*

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