Let $n \in \mathbb{N}$. Then, the circulant Latin square $C^{(n)}$ contains a **Latin transversal** if and only if n is odd.

$$
\begin{array}{c|cccccc}
+ & & & & & & \\
\hline
 & 0 & 1 & 2 & \cdots & & n-1 \\
 & 1 & 2 & 3 & \cdots & n-1 & 0 \\
 & & & & \vdots & & \\
 & n-1 & 0 & 1 & \cdots & & n-2
\end{array}
$$

Proof. If n is odd, then its diagonal are of distinct entries $0, 2, \cdots, n-1, 1, 3, 5, \cdots, n-2$, hence a Latin transversal. On the other hand, if n is even. Suppose that we can find a Latin transversal with entries $l_{0,i_0}, l_{1,i_1}, l_{2,i_2}, \cdots, l_{n-1,i_{n-1}}$.

Then, $\displaystyle\sum_{j=0}^{n-1} l_{j,i_j} = \sum_{i=0}^{n-1} i = \frac{n(n-1)}{2} \equiv \frac{n}{2} \pmod{n}$. Now, by using the additive group $\langle \mathbb{Z}_n, + \rangle$, $l_{j,i_j} = j + i_j$.

Therefore, $\displaystyle\sum_{j=0}^{n-1} l_{j,i_j} = \sum_{j=0}^{n-1} (j + i_j) = 2 \cdot \frac{n(n-1)}{2} \equiv 0 \pmod{n}$. $\rightarrow\leftarrow$

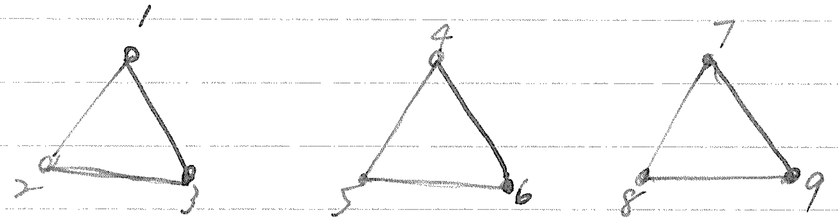
Hence, it is not possible to find such a Latin transversal.

$\blacksquare$

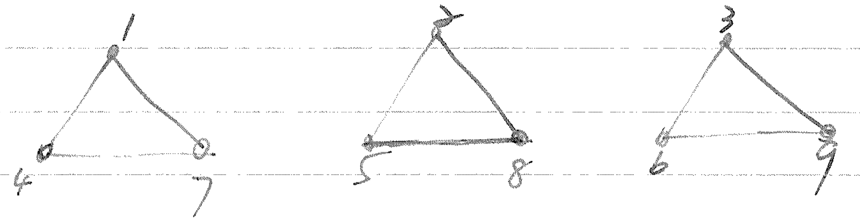
(e) $n = 10$, Not possible!

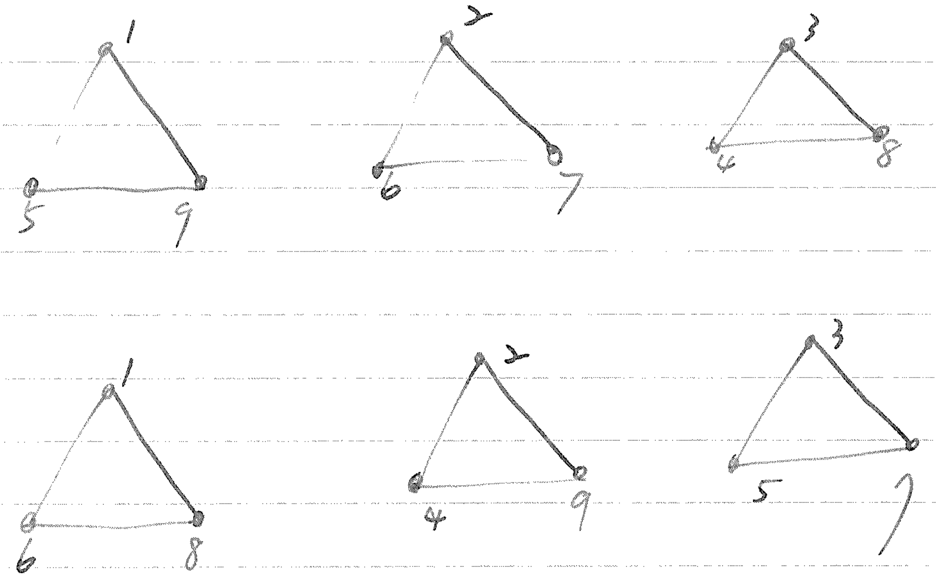
Lecture 13 Combinatorial Designs 6,8

Date / /   No. 13-1



安排9個人
坐在三角形桌
一起吃飯;
4天後任兩個
人都同桌過。

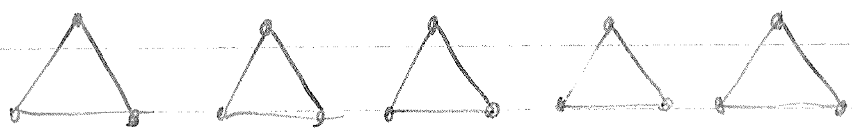




1 2 3	1 4 7	1 5 9	1 6 8
4 5 6	2 5 8	2 6 7	2 4 9
7 8 9	3 6 9	3 4 8	3 5 7

(Correct answer?)

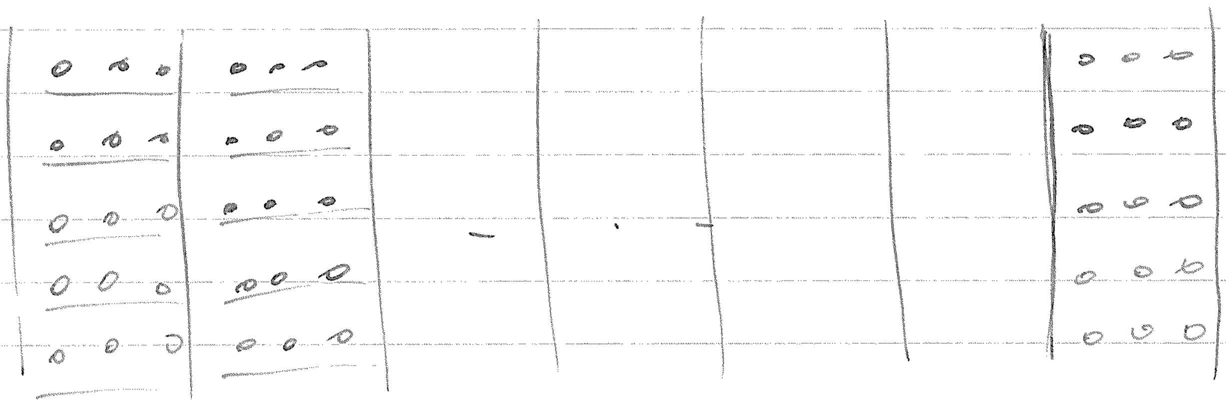
How about 1~15, five triangular tables?



How about 16 people sitting on 4 people tables?

Kirkman's 15 school girls problem

15個女學生每天排路隊上學，分成五列，每列三人；同一列的三人会互相認識；要如何安排最少上學的日子，讓她们彼此都互相認識。(?)



(1) 每天只可能認識其他的兩位，因此至少要"7"天才能完成這個安排的工作。

(2) 每天只有可能有 15 对互相認識的同學，所以最好的安排是恰好 $\binom{15}{2}/15 = 7$(天)。

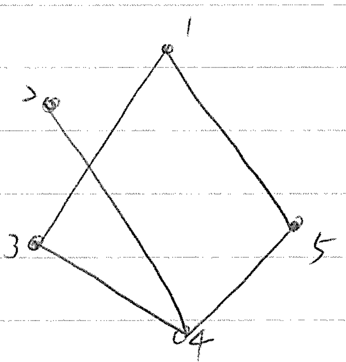
以下是一个可能的答案。

1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	1 14 15
4 8 12	2 8 10	2 9 11	2 12 15	2 13 14	2 4 6	2 5 7
5 10 14	3 13 15	3 12 14	3 5 6	3 4 7	3 9 10	3 8 11
6 11 13	6 9 14	4 10 15	4 11 14	5 9 12	5 11 15	4 9 13
7 9 15	7 11 12	5 8 13	7 10 13	6 8 15	7 8 14	6 10 12

Definition (Combinatorial Design)

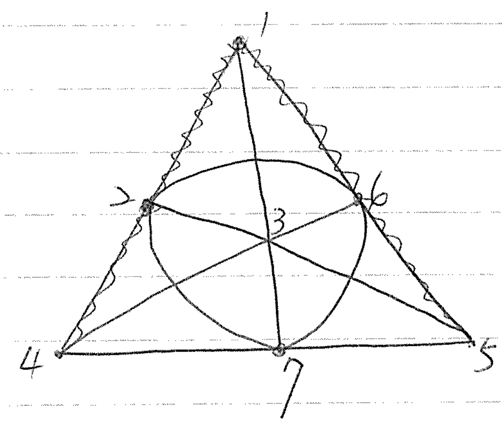
A design is an ordered pair $(X, \mathbb{B})$ where X (Varieties) is a non-empty set and $\mathbb{B}$ is a collection of subsets (or multi-sets) of X. (Blocks)

e.g. $X = \{1,2,3,4,5\}$, $\mathbb{B} = \{\{1,3\},\{1,5\},\{2,4\},\{3,4\},\{4,5\}\}$.



(∘) If $\mathbb{B}$ is a collection of 2-subsets, then $(X, \mathbb{B})$ is in fact a graph.

e.g. $X = \{1,2,3,4,5,6,7\}$, $\mathbb{B} = \{124, 235, 346, 457, 561, 672, 713\}$ is also a design.



Fano plane

(∘) If the subsets in $\mathbb{B}$ have more than two elements, then we can use "lines" to represent the sets.

(∘∘) 所以設计也可以看成"有限几何"。

(∘∘∘) 在圖论中，边由多於两点所構成的圖為 Hypergraph。
(超圖)

<u>Goal</u>: A design with better properties!

(1) A design $(X, \mathbb{B})$ is an <u>incomplete</u> design if $\forall B \in \mathbb{B}, |B| < |X|$.

(2) A design $(X, \mathbb{B})$ is balanced if $\forall v \in X$, v occurs in "r" blocks of $\mathbb{B}$.

(3) A design $(X, \mathbb{B})$ is pairwise balanced if for any pair of two varieties x and y, they occur together in "λ" blocks.

(4) We can represent $(X, \mathbb{B})$ by a $(0,1)$-matrix, called incidence matrix of $(X, \mathbb{B})$. Let $X = \{x_1, x_2, \cdots, x_v\}$ and $\mathbb{B} = \{B_1, B_2, \cdots, B_b\}$. Then, the incidence matrix of $(X, \mathbb{B})$ is

$$A: \quad \begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ x_i \\ \\ x_v \end{array} \begin{array}{c} B_1 \quad B_2 \quad - \quad B_j \quad\quad B_b \\ \left[\begin{array}{ccccc} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{array} \right] \end{array} \qquad A(i,j) = \begin{cases} 1, & \text{if } x_i \in B_j, \\ 0, & \text{otherwise} \end{cases}$$

Date / /

e.g. $X = \{1, 2, \cdots, 9\}$, $\mathbb{B} = \{\overset{B_1}{123}, \overset{B_2}{456}, \overset{B_3}{789}, \overset{B_4}{147}, \overset{B_5}{258}, \overset{B_6}{369}, \overset{B_7}{159}, \overset{B_8}{267}, \overset{B_9}{348}, \overset{B_{10}}{168}, \overset{B_{11}}{249}, \overset{B_{12}}{357}\}$.

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccc} 123 & 456 & 789 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 0 & \\ 1 & 0 & 0 & \\ 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ 0 & 0 & 1 & \\ 0 & 0 & 1 & \end{array} \right] \end{array}_{9 \times 12}$$

$\boxed{1 \mid 4 \mid 7 \mid 10}$ ( ) $\boxed{1 \mid 5 \mid 8 \mid 11}$ ( ) $\boxed{1 \mid 6 \mid 9 \mid 12}$ ( ) $\boxed{2 \mid 4 \mid 9 \mid 11}$ ( )

$\boxed{2 \mid 5 \mid 7 \mid 12}$ ( ) $\boxed{2 \mid 6 \mid 8 \mid 10}$ ( ) $\boxed{3 \mid 4 \mid 8 \mid 12}$ ( ) $\boxed{3 \mid 5 \mid 9 \mid 10}$ ( )

$\boxed{3 \mid 6 \mid 7 \mid 11}$ ( )

~~1~~, ~~2~~, ~~3~~, ~~4~~, 5, ~~6~~, ~~7~~, 8, ~~9~~, ~~10~~, ~~11~~, ~~12~~

This matrix (array) can be applied in Group Testing.

(Guess two numbers in your mind by answering the above questions (queries)).

What is "Group Testing"?

N : A set of n items.

D : A subset of N, called defective items.
(defectives)

Test : <u>Ask</u> a subset of N, S, to check if

(Negative) $S \cap D = \phi$ or

(Positive) $S \cap D \neq \phi$.

Goal : Find the set of defectives, D.

(·) $|D| \lll |N| = n$.

=

<u>Binary</u> <u>Spliting</u> <u>Algorithm</u> : Use at most $d \cdot \lceil \log_2 n \rceil$ tests.

<u>Idea</u> Find one defective in $\lceil \log_2 n \rceil$ tests.

<u>Problem</u> If $d \geq 2$, can we find an algorithm to obtain D with less than $d \lceil \log_2 n \rceil$ tests.

<u>e.g.</u> $n = 56$, $d = 2$, Can we find the defectives in "11" tests?