

Lecture 12

NO. /

DATE

(t-design)

When $t \geq 3$, then finding a good t -design is getting more complicated especially when the block size is also larger. The followings are some basic properties of t -designs.

• If (X, \mathcal{B}) is a t - (v, k, λ) design, then

(a) $\lambda \binom{v}{t} / \binom{k}{t}$ is an integer,

(b) for each $0 \leq i \leq t$, the collection of all blocks B_i

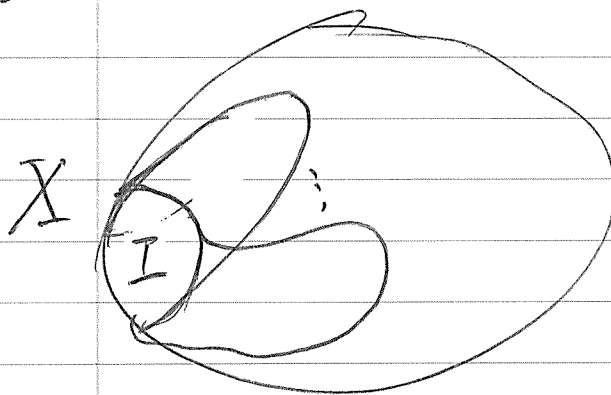
containing a given i -subset of X is exactly

$$\lambda \binom{v-i}{t-i} / \binom{k-i}{t-i}, \text{ and}$$

(c) if I is an i -subset with $i \leq t$, then the collection

of blocks $\mathcal{B}_i = \{B \cap I \mid B \in \mathcal{B}\}$ is a $(t-i)$ - $(v-i, k-i, \lambda)$
with $X_i = X - I$

design.



• Let $k \geq t \geq 2$. Then, the collection of all k -subsets of $X = \mathbb{Z}_v$ is in fact a t -design t - (v, k, λ) design where
 $\lambda = \binom{v-t}{k-t}$ if $k > t$ and $\lambda = 1$ if $k = t$.

• If $k=3$ and $t=2$, then $\binom{\mathbb{Z}_v}{3}$ forms a 2 - $(v, 3, \lambda)$ design where $\lambda = v-2$. (If $k=4$ and $t=2$, then $\binom{\mathbb{Z}_v}{4}$ is a 2 - $(v, 4, \lambda)$ design with $\lambda = \binom{v-2}{2}$.)

• In case of $k=3$, if $\binom{\mathbb{Z}_v}{3}$ can be partitioned into $v-2$ disjoint STS(v)'s, then we have a large set of Steiner triple systems.

(*) You may try the case when $v=7$ and $k=3$.

• Theorem (Lu, Jia-Xi, 陸嘉義) 1935-1983 高中老師

A large set of STS(v)'s exists except for some small cases.

Definition (Steiner systems)

In a t -design (X, \mathcal{B}) , if $k = t+1$ and $\lambda = 1$, Then we have a Steiner t -design of order v of order $|X|$. A Steiner triple system is a 2 - $(v, 3, 1)$ design and a Steiner quadruple system of order v is a 3 - $(v, 4, 1)$ design or $S(t, k, v)$ in short where $t = k-1$.

- Small examples $t=3$ and $k=4$

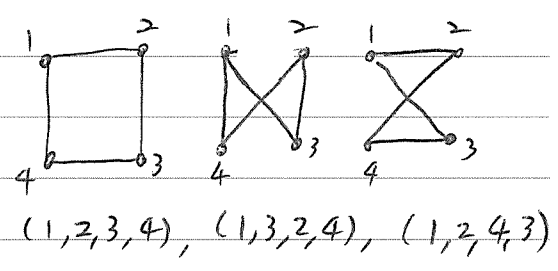
Let $X = \mathbb{Z}_2^4$ and $B = \{ \vec{w}, \vec{x}, \vec{y}, \vec{z} \} \mid \vec{w}, \vec{x}, \vec{y}, \vec{z} \in \mathbb{Z}_2^4, \vec{w} + \vec{x} + \vec{y} + \vec{z} = \vec{0} \}$

Notice that $\vec{x}, \vec{y}, \vec{z}$ and \vec{w} are distinct vectors. Then, (\mathbb{Z}_2^4, B) is

an $S(3, 4, 16)$. It is also true for an $S(3, 4, 2^m)$ where $m \in \mathbb{N}$.

- Let $X = E(K_5)$ and $B = \left\{ \begin{array}{c} \triangle \\ (5) \\ (1) \end{array}, \begin{array}{c} \triangle \\ \longrightarrow \\ (5) \\ (3) \end{array}, \begin{array}{c} \square \\ (5) \\ (4) \end{array} \right\} \Bigg| \begin{array}{l} \text{Labeled} \\ \text{subgraphs of } K_5 \end{array}$

Then, (X, B) is an $S(3, 4, 10)$.



- How about $t=3$ and v in general?
- $v \equiv 2$ or $4 \pmod{6}$. (Let (X, B) be an $S(3, 4, v)$.)

Let $x_0 \in X$ and $B' = \{ B \setminus \{x_0\} \mid B \in B \}$. Then, (X', B') is an STS($v-1$),
 $(X' = X \setminus \{x_0\})$
 This implies that $|X'| = v-1 \equiv 1$ or $3 \pmod{6}$.

- Theorem (H. Hanani, 1960)

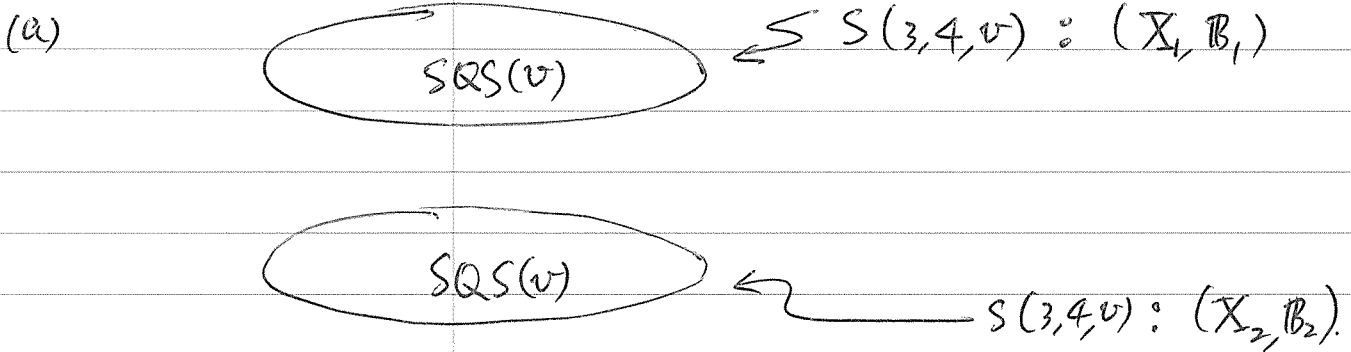
An $S(3, 4, v)$ exists if and only if $v \equiv 2$ or $4 \pmod{6}$.

Proof. It takes a lot of effort in proving the sufficient part.

Doubling Construction

Construct 1. An $S(3,4,2v)$ exists if an $S(3,4,v)$ exists.

Proof.



Let K_{X_i} denote the complete graph defined on X_i , $i=1,2$.

Since $|X_i|$ is even, K_{X_i} can be decomposed into 1-factors, there

are $v-1$ of them, called F_1, F_2, \dots, F_{v-1} and G_1, G_2, \dots, G_{v-1}

for $i=1,2$ respectively. Now, we use F_j and G_j , $j=1,2,\dots,v-1$

to define $\binom{v}{2}$ quadruples by the following way:

$$F_j = \{ \{a_i, b_i\} \mid i=1,2,\dots,\frac{v}{2} \} \Rightarrow \{a_i, b_i, c_j, d_j\} \in \mathcal{B}$$

$$G_j = \{ \{c_j, d_j\} \mid j=1,2,\dots,\frac{v}{2} \}$$

Combining with \mathcal{B}_1 and \mathcal{B}_2 , we have an $S(3,4,2v)$.

(or $SQS(2v)$).

(*) This $SQS(2v)$ contains two disjoint sub-designs $SQS(v)$.

(b) Let $Y' = \{y' \mid x \in Y\}$ and $X = Y \cup Y'$. Let (Y, \mathcal{C}) be an SQS(v).

Define \mathcal{B} .

(1) $\forall \{x, y, z, w\} \in \mathcal{C}$, let $\{x, y, z, w\}, \{x, y, z', w\}, \{x, y', z, w\}, \{x', y, z, w\}, \{x', y', z, w\}, \{x', y, z', w\}$ and $\{x, y', z', w\}$ be in \mathcal{B} .

(2) For any two elements $\{x, y\} \in Y$, let $\{x, y, x', y'\} \in \mathcal{B}$.

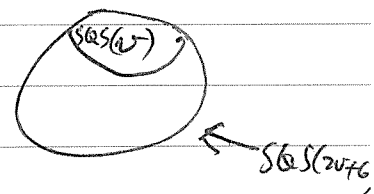
Combining (1), (2), we have an SQS($2v$) = (X, \mathcal{B}) .

It is a routine matter to check (X, \mathcal{B}) is indeed an SQS($2v$) for both constructions: (a) and (b).

The above doubling construction can only handle the cases $v \equiv 4$ or $8 \pmod{12}$. For the other cases, it takes more effort.

(We omit the details here.)

Conjecture $v \rightarrow 2v+6$ Construction
 (X_v, \mathcal{B}_v)



For each SQS(v), there exists an SQS($2v+6$) which contains (X_v, \mathcal{B}_v) as a subsystem.

(*) If we have $v \rightarrow 2v$ and $v \rightarrow 2v+6$ constructions, then the theorem about the existence of SQS(v)'s is proved.

Proof. Consider $v \equiv 2$ or 4 or 8 or $10 \pmod{12}$. Clearly, if $v \equiv 4$ or $8 \pmod{12}$, then by $v \rightarrow 2v$, we can construct such a system. On the other hand, if $v \equiv 2$ or $10 \pmod{12}$, let $v = 12k+2$ or $12k+10$ respectively. By direct counting, $12k+2 = 2(6k-2)+6$ and $12k+10 = 2(6k+2)+6$. Hence, the construction $v \rightarrow 2v+6$ works. \blacksquare

The best known construction besides $v \rightarrow 2v$ or SQS(v) is the following

Theorem (Hartman) (Tripling Construction!)

If an SQS(v) contains a subsystem SQS(u), then there exists an SQS($3v-2u$) which contains ^{the above} SQS(v), ...

Note that we can also use this theorem to prove the cases

$v \equiv 2$ or $10 \pmod{12}$, (?)
except some small cases.

$$v \equiv 2 \text{ or } 10 \pmod{12}$$

$$\Rightarrow v \equiv 2, 10, 14, 22, 26, 34 \pmod{36}$$

$$36k+2 = 3 \cdot (12k+2) - 4 \quad (u=2) \quad \text{SQS}(2) \text{ is a trivial system}$$

$$36k+10 = 3(12k+4) - 2 \quad u=1 \text{ (one element)}$$

$$36k+14 = 3(12k+10) - 16 \quad u=8$$

$$36k+22 = 3(12k+14) - 20 \quad u=10$$

$$36k+26 = 3(12k+14) - 16 \quad u=8$$

$$36k+34 = 3(12k+14) - 8 \quad u=4$$

Exercise 2.7. (20 points)

Constructing SQS(v) for as many v as possible.

(* Turn in your Homework (II) by combining 120 points.
(at most)