Partial Ordered Sets (Poset)

Definition

A relation defined on a set X, $<$, is called a partial order of X if

(i) $\forall\ a \in X$, $a < a$,

(ii) $\forall\ a, b \in X$, if $a < b$ and $b < a$, then $a = b$, and

(iii) $\forall\ a, b, c \in X$, $a < b$ and $b < c$ imply $a < c$.

e.g. In $\mathbb{R}$, "$\leq$" is a partial order of $\mathbb{R}$.

e.g. Let $X = \{1, 2, \cdots, n\} = [n]$. The set inclusion "$\subseteq$" is a partial order of X.

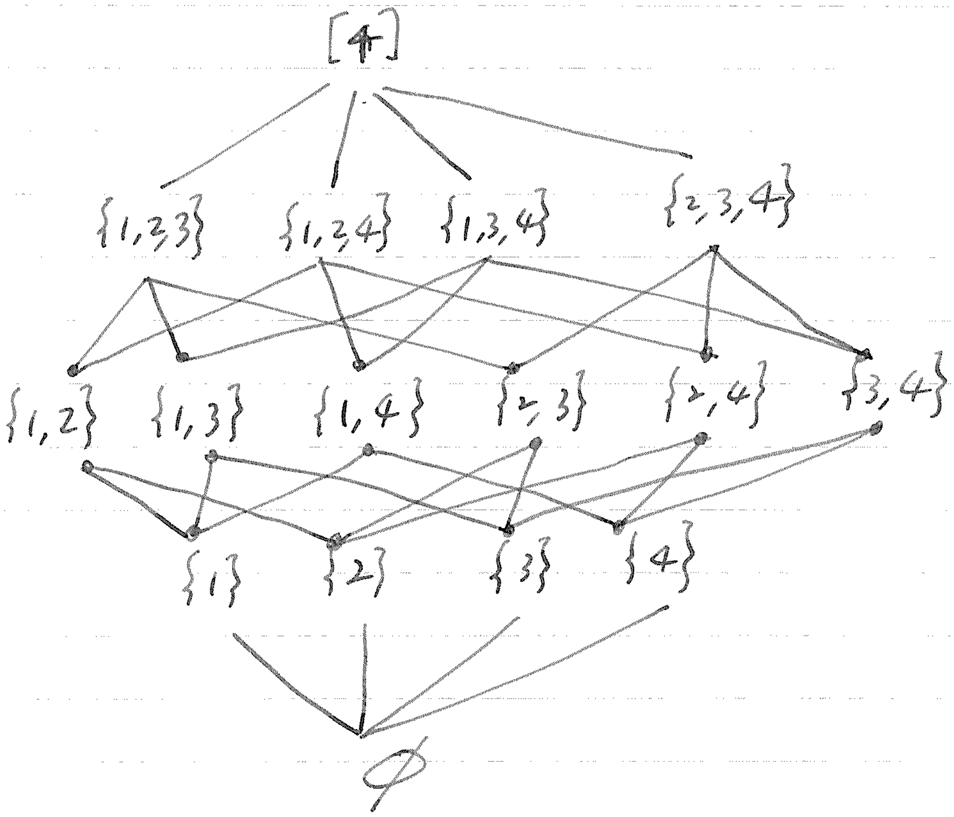
(∘) A partial order "$<$" of X is a total order if $\forall\ a, b \in X$ either $a < b$ or $b < a$.

(∘∘) "$\subseteq$" in $[n]$ is not a total order since for example $\{1, 2, 3\} \not\subseteq \{1, 2, 4\}$ and $\{1, 2, 4\} \not\subseteq \{1, 2, 3\}$.

(*) It is more interesting in considering a partial order which is not a total order.

For clearness, we may use a figure (Hasse Diagram) to depict the relationship between sets.

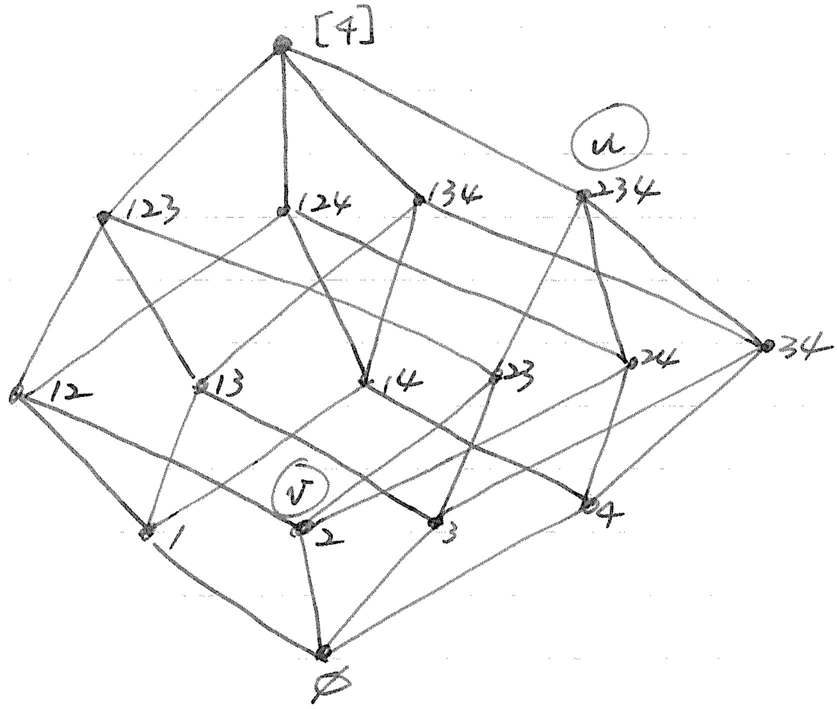
Let $n=4$.



(*) 同一層的集合沒有互相包含的關係。

(*) 不同層的集合"可能"有包含關係，如果在上層的集合包含下層的集合。

(*) 上圖可以改用 Graph 來表示，見下頁。



(∘) 由上而下，如果有一條路徑可以連接，則上面的集合包含下面的集合，如上圖的 u, v.

(∘) 這樣"Path"又稱為 Chain.

(∘∘) 不互相包含的所有集合，形成一個 Anti-chain.

例如： 12, 13, 14, 23, 24, 34.

Theorem (Sperner)

Consider the collection of all subsets of $[n]$. The maximum number of subsets which do not contain each other (anti-chain) is equal to $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Proof. Since $\binom{[n]}{\lfloor \frac{n}{2} \rfloor}$ (the sets with exactly $\lfloor \frac{n}{2} \rfloor$ elements) is an anti-chain, the maximum number $\alpha_n \geq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Now, we claim that $\alpha_n \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Let A be a collection of subsets which attain the maximum.

Consider the set of all permutation of $[n]$. Clearly, there are $n!$ permutations. Now, for each set S in A, we associate this set with $|S|!(n-|S|)!$ permutations

$$\alpha = \left(\begin{array}{c|c} S & [n]\setminus S \\ S & [n]\setminus S \end{array}\right).$$

(For example, if $n=6$ and $S=\{1,3,4\}$, then

$$\alpha = \left(\begin{array}{c|c} 1\ 3\ 4 & 2\ 5\ 6 \end{array}\right).)$$

$$\begin{array}{cc} 1,3,4 & 2,5,6 \end{array}$$

On the other hand, each permutation in S_n can associate with at most one set in A.

Suppose not. Let β be a permutation associates with S_1 and S_2 in A. For S_1,

$$\beta = \left(\begin{array}{c|c} S_1 & [n]\setminus S_1 \\ S_1 & [n]\setminus S_1 \end{array}\right).$$

Hence, if β associates S_2, then either $S_2 \subseteq S_1$ or $S_1 \subseteq S_2$, a contradiction.

Thus, $\sum_{S\in A} |S|!\,(n-|S|)! = \sum_{k=0}^{n} a_k \cdot k!\,(n-k)! \le n!$.

(a_k is the # of sets in A with k elements.)

$$\Rightarrow \sum_{k=0}^{n} a_k \cdot \frac{k!\,(n-k)!}{n!} \le 1$$

$$\Rightarrow \sum_{k=0}^{n} a_k \Big/ \binom{n}{k} \le 1 \quad \text{(Lubell-Yamamoto-Mashalkin)}$$

$$\Rightarrow \sum_{k=0}^{n} a_k \Big/ \binom{n}{\lfloor \frac{n}{2} \rfloor} \le 1 \Rightarrow \sum_{k=0}^{n} a_k \le \binom{n}{\lfloor \frac{n}{2} \rfloor}. \quad \blacksquare$$

以下的定理讨论如何找最多的r元子集([n]),使得它们之间,两两都相交,亦即 $\forall\, S_1, S_2 \in B,\ S_1 \cap S_2 \neq \emptyset$.

Consider $n=6$, $r=3$.

In total, there are $\binom{6}{3} = 20$ subsets.

$\boxed{123, 124, 125, 126, 134, 135, 136, 145, 146, 156}$, 234, 235, 236

245, 246, 256, 345, 346, 356, 456.

(*) How to choose subsets such that they are mutually intersecting? And how many?

→ A collection of <u>such</u> <u>family</u>! 10 of them.
(Can we find more than 10?)

<u>Definition</u> A collection of r-subsets of $[n]$ is called an r-uniform interesting family if any two sets in the family are interesting, i.e., their intersection is non-empty. This family with maximum number of sets is denoted by $B_{n,r}$.

<u>Theorem</u> $|B_{n,r}| = \binom{n-1}{r-1}$ $\forall\, n \in \mathbb{N}$ and $n \geq 2r$. (Erdös-Ko-Rado Thm.) EKR Thm.

<u>Proof</u>. By taking a fixed common element n and the other $r-1$ elements from $[n-1]$, we conclude that $|B_{n,r}| \geq \binom{n-1}{r-1}$.

It is sufficient to claim that $|B_{n,r}| \leq \binom{n-1}{r-1}$.

Let $(a_1, a_2, \cdots, a_n)$ be a cyclic arrangement of $[n]$.
Let A be an r-uniform intersecting family.
Then, following the order of cycle, this cycle contains at most r sets of A. (For example, consider $n=8$ and $r=3$. Let $(3, 1, 8, 2, 7, 5, 6, 4)$ be a cyclic arrangement. Now, if $\{8, 2, 7\}$ is a set in A, then we have two more along the cycle, namely $\{1, 8, 2\}$ and $\{2, 7, 5\}$.)

Since there are $(n-1)!$ cyclic arrangements using n elements, we have at most $r \cdot (n-1)!$ sets in A..

Now, we use two way counting to count the number of pairs (S, α) where $S \in A$ and α is a cyclic arrangement. By a similar idea in proving Sperner's Theorem, for each S in A, we can generate $r!(n-r)!$ cyclic arrangements. (For example, $\{8,2,7\}$ in the above example, we have $3! \cdot 5!$ different arrangements.

8, 2, 7,	1, 3, 4, 5, 6
8, 7, 2	↓
2, 7, 8	
2, 8, 7	
7, 8, 2	
7, 2, 8	(5!)
(3!)	

Moreover, these arrangements can have $|A| \cdot r!(n-r)!$ pairs (S, α). On the other hand, for each cyclic arrangement, there are at most three sets in A, hence

$$|A| \cdot r!(n-r)! \leq r \cdot (n-1)!.$$

This concludes, the proof that $|A| \leq \binom{n-1}{r-1}$. ■

(*) 左式比較小，是因為每一个 pair (S, α) 都可以在 $r \cdot (n-1)!$ 个 arrangements 找到。

最多 r 个