Lecture 10

We can also use MOLS(n) to construct PBDs in which K is of size larger than one. For example, we can use an Affine plane of order 5 to construct a PBD $2-(24,\{4,5\},1)$ design: $(\mathbb{X},\mathbb{B})$. The idea comes from deleting an element from $\mathbb{X}$. Then, each block which contains this element becomes a block of size 4, and the other blocks which do not contain this element remain the same.

Hence, we can start with a special type of design, and then either adding or deleting elements (to or from) $\mathbb{X}$ to obtain a new design.

<u>Definition</u> (Group Divisible Designs of type n^m)

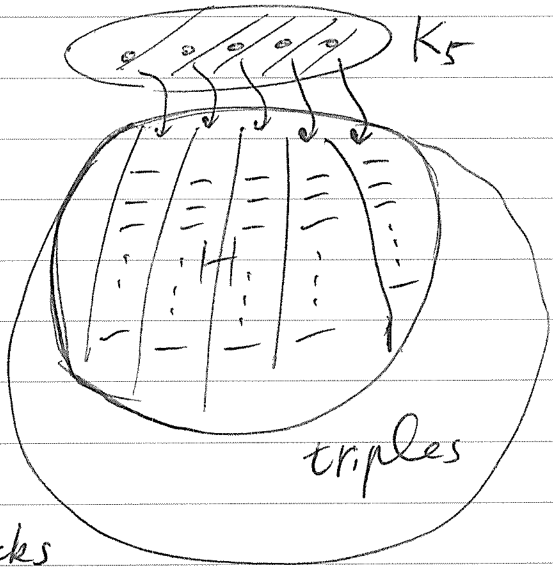
A design $(\mathbb{X},\mathbb{B})$ is called a group divisible design of type n^m if $\mathbb{X}$ can be partitioned in m disjoint subsets $G_1, G_2, \cdots, G_m$ (called groups) such that each $B \in \mathbb{B}$, $|B \cap G_i| \le 1$, $|B| = k$ and every pair of two elements from different groups occurs together in exactly λ blocks of $\mathbb{B}$.

The design $(\mathbb{X},\mathbb{B})$ is denoted by $GDD(n,m;k;\lambda)$.

A $GDD(n,m;k;\lambda)$ can be shorten as a k-GDD of type n^m and index λ. We shall solve the case $k=3$ and $\lambda=1$ in what follows. First, we need a theorem.

<u>Theorem 2.</u> For each $v \equiv 5 \pmod 6$, a 2-$(v, \{3,5\}, 1)$-design exists. Moreover, we have such a design with exactly one block of size 5.

<u>Proof.</u> (By difference method.) Let $v = 6k+5$ and $X = X_1 \cup X_2$ where $|X_1| = 5$ and $|X_2| = 6k$. Now, let $X_2 = \mathbb{Z}_{6k}$. Hence, the set of differences in $\mathbb{Z}_{6k} = \{1, 2, \cdots, 3k \text{(half)}\}$. As mentioned in the above construction, we can find difference triples either in $\{1, 2, \cdots, 3k-3\}$ or $\{1, 2, \cdots, 3k-\cancel{4}, 3k-2\}$. Hence, after taking away those triples, we have a 5-regular graph H left defined on $\mathbb{Z}_{6k}$. Since $3k$ is one of the differences, $\chi'(H) = 5$. The proof then follows by the same idea as in recursive construction.



<u>Note.</u> Such a PBD also exists for $v \equiv 1$ or $3 \pmod 6$ since we can take all blocks of size 3.

Group Divisible Design (3 – GDD)

Problem For which m and n, $K_3 \mid K_{m(n)}$?

Fact 1. If $n=1$, then $m \equiv 1$ or $3 \pmod 6$.

Definition (3-sufficient)

A graph G is said to be 3-sufficient if (1) $|G| \geq 3$, (2) G is an even graph and (3) $3 \mid \|G\|$.

Problem (Open) For which 3-sufficient graph G, $K_3 \mid G$?

Nash-Williams Conjecture (Remains open)

If G is 3-sufficient and $\delta(G) \geq \frac{3}{4}|G|$, then $K_3 \mid G$.

Fact 2. If $K_{m(n)}$ is 3-sufficient, then

(1) Either n is even or n is odd and m is odd; and

(2) $3 \mid \binom{m}{2} \cdot n^2$.

Theorem. If $K_{m(n)}$ is 3-sufficient and $m \geq 3$, then $K_3 \mid K_{m(n)}$

We need several basic facts in order to prove the theorem

Fact 3. $K_3 \mid K_{3(n)}$. (By using a L.S. of order n.)

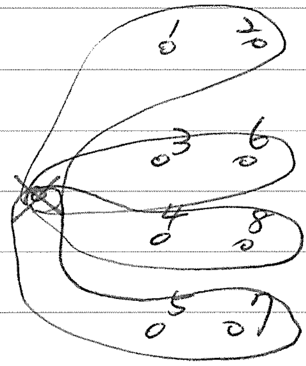
Fact 4. $K_3 \mid K_{4(n)}$ if and only if n is even.

Proof. ($\Rightarrow$)

Since $m=4$, n must be even in order that each vertex is of even degree.

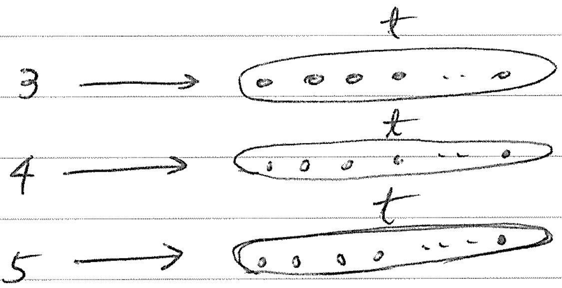
($\Leftarrow$) If $n=2$, then $K_3 \mid K_{4(2)}$. This is a consequence of deleting one vertex of an STS(9). (See it?)

012	036	048	057
345	147	156	138
678	258	237	246



Now, let $n=2t$. The proof follows by blowing up each vertex into t vertices and use an LS(t) to construct all the K_3's we need.

For example,



As a consequence, we have $8 \cdot t^2$ K_3's in total. This is also the number K_3's we desire: $\frac{6 \cdot (2t)^2}{3} = 8t^2$.

<u>Fact 5.</u> If $m \equiv 1$ or $3 \pmod 6$, then $K_3 \mid K_{m(n)}$ for each positive integer n.

Proof. It is a direct consequence of blowing each vertex of K_m into n vertices. ■

Fact 6. If $m \equiv 0$ or $4 \pmod 6$ and n is even, then $K_3 \mid K_{m(n)}$.

Proof. First, we take an $STS(2m+1)$ $(X, \mathcal{B})$ and delete one vertex from X, then we have $K_3 \mid K_{m(2)}$. Since n is even, we use the same technique as that in Fact 4. The concludes the proof. ■

Fact 7. If $m = 5$ and $3 \mid n$, then $K_3 \mid K_{m(n)}$.

Proof. Let $n = 3k$. By the fact that $K_3 \mid K_{5(3)}$, we conclude the proof by blowing each vertex into k vertices. ■

Fact 8. If $m \equiv 5 \pmod 6$ and $3 \mid n$, then $K_3 \mid K_{m(n)}$.

Proof. This is a direct result of the existence of a PBD $(m, \{3, k\}, 1)$-design and Fact 7. ■

Fact 9. If $m \equiv 2 \pmod 6$ and $6 \mid n$, then $K_3 \mid K_{m(n)}$.

Proof. Let $m = 6k+2$. Consider $2m+1 \equiv 5 \pmod 6$. Since a $(2m+1, \{3,5\}, 1)$-design exists, we may let it be as in the following figure.

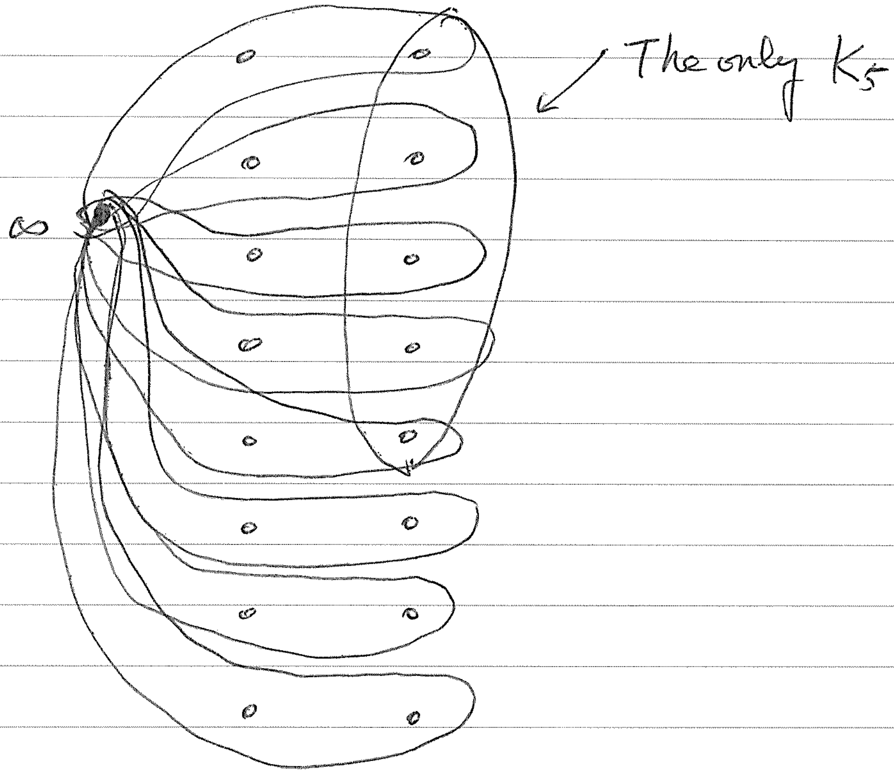


Figure for $(2m+1, \{3,5\}, 1)$ PBD

Now, by deleting ∞, we obtain a decomposition of $K_{m(2)}$ into K_3's and one K_5. Let $n = 6k$. Then, the proof follows by blowing up each vertex into $3k$ vertices. ∎

Theorem (3-GDD)

$K_3 \mid K_{m(n)}$ if and only if $K_{m(n)}$ is 3-sufficient.

Proof. Combining Facts 5, 6, 7, 8, 9; we have the proof. ∎

Exercise 2.5. (20 points) Prove the above theorem in details.