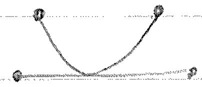
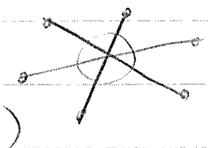
May 15, 18

 Lecture 10 Euler's Planar Graph Theorem

Drawing a graph on plane (sphere)

- Two edges intersect at "at most" one point.
- Each edge is represented by a closed simple curve.
- The point in which two edges intersect is a "crossing".
- Tangent type of intersection is forbidden. 
- No multiple (≥ 2) crossings 

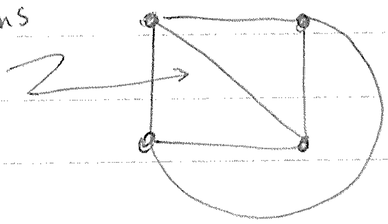
Definition (Crossing number)

The crossing number of a graph G is the minimum number of crossings in all possible drawings, denoted by $cr(G)$.

Definition (plane graph or planar graph)

The graph G with $cr(G) = 0$ is called a plane graph if the graph is drawn on a plane and it's planar if the drawing is on a sphere. (Notice that a plane graph has an outer-region and no outer-region for a planar graph.)

regions



A planar graph

Theorem (Euler)

If G is a connected planar graph with p vertices, q edges and r regions, then $p-q+r=2$.

Proof. By induction on the number of edges.

Since G has at least $p-1$ edges, the induction starts at $q=p-1$. In this case, G is a tree and G has exactly one region. Hence, $p-(p-1)+1=2$, the assertion is true. Now, assume that the assertion is true for $q=k\geq p-1$.

Consider G where $|E(G)|=k+1$ and G has r regions.

Since $|E(G)|=k+1\geq p$, G contains a cycle. Let e be an edge which is on a cycle. Clearly, $G-e$ (delet the edge e) has exactly k edges and $G-e$ is still connected. Hence, in $G-e$, it has p vertices, $\overset{k}{(q-1)}$ edges and $r-1$ regions. By induction $p-(q-1)+(r-1)=2$ and thus

$p-q+r=2$. We have the proof. ■

(o) This equality is a necessary condition for the existence of a planar graph.

(◦◦) In general, it is not easy to count the number of regions after drawing the graph properly on a surface.

Fact If a planar graph is of maximum size, then all its regions are triangles (or called triangular faces).

Fact 1. If G is a connected (p,q)-graph, then $q \leq 3p-6$.

Proof. Since all regions are triangles, $3r = 2q$. By Euler's formula $p - q + \frac{2}{3}q = 2$. $\frac{1}{3}q = p-2 \Rightarrow q = 3p-6$. ∎

Fact 2. K_5 is not a planar graph since $q = 10 > 3\cdot 5 - 6$.

Fact 3. If the minimum length of cycles in a planar graph G ((p,q)-graph) is $g \geq 3$, then $q \leq \frac{g}{g-2}(p-2)$.

Fact 4. $K_{3,3}$ is not a planar graph.

Proof. The minimum length of a cycle in $K_{3,3}$ is 4. Hence, $q \leq \frac{4}{4-2}(6-2) = 8$ if $K_{3,3}$ is a planar graph. But, $K_{3,3}$ has 9 edges. ∎

Fact 5. The Petersen graph is not a planar graph.

Proof. $g = 5$ in Petersen graph P. If P is planar, then $q \leq \left\lfloor \frac{5}{5-2}(10-2) \right\rfloor = 13$. ∎

(*) Lemma If G is a (p,q)-planar graph, then there exists a vertex of G whose degree is at most "5".

Proof. If every vertex of G is of degree at least 6, then G has at least $3p$ edges. But, $q \leq 3p-6$, a contradiction. ■

(∘) If G is a regular polyhydron (正多面体), then G is a regular planar graph. Moreover, each vertex is of degree 3, 4, or 5. (?)

(∘∘) The faces are 3-face, 4-face and 5-face respectively. (?)

(∘) Let p_k and r_k denote the number of vertices of degree k and k-faces respectively.

(∘) $r_k \cdot k = 2q \Rightarrow q = \frac{1}{2} k \cdot r_k$.

Theorem There are exactly five regular polyhedra.

Proof. $p - q + r = 2$

$$-8 = 4q - 4p - 4r = \underline{2q} + 2q - 4p - 4r$$

$$= h \cdot r_h + k \cdot p_k - 4p_k - 4r_h = (k-4)p_k + (h-4)r_h$$

Since $3 \leq h, k \leq 5$, we have nine cases to check.

四面体

Case 1 : $h=3,\ k=3$. $-8=-r_3-p_3$, $3r_3=3p_3 \Rightarrow r_3=p_3=4$ (Tetrahedron)

Case 2 : $h=3,\ k=4$. $r_3=8,\ p_4=6$, Octahedron, 八面体.

Case 3 : $h=3,\ k=5$. $r_3=20,\ p_5=12$, Icosahedron, 二十面体.

Case 4 : $h=4,\ k=3$. $r_4=6,\ p_3=8$, Cube, 六面体

Case 5 : $h=4,\ k=4$. $-8 \neq 0$ (Not possible!)

Case 6 : $h=4,\ k=5$. $-8 = p_5$ (N.p.)

Case 7 : $h=5,\ k=3$. $r_5=12,\ p_3=20$, Dodecahedron 十二面体

Case 8 $h=5,\ k=4$. $-8 \neq r_5$ (N.P.)

Case 9 $h=5,\ k=5$. $-8 \neq r_5+p_5$ (N.P.) ∎