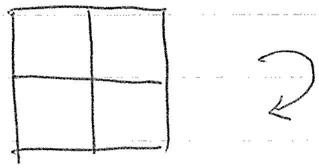
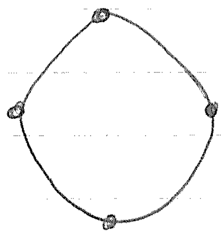
Date # 補充資料





可以看成項鍊，但是只考慮旋轉。

Cycle index $P_G(z_1,z_2,z_3,z_4) = z_1^4 + 2z_4 + z_2^2$

塗兩色：$P_G(2,2,2,2)/4 = (16+4+4)/4 = 6$

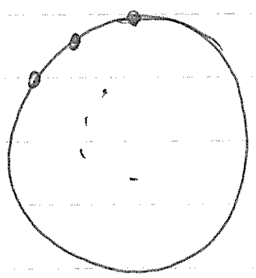
塗三色：$P_G(3,3,3,3)/4 = (81+6+9)/4 = 24$.

(*) 可以利用只有旋轉的狀況來證明費馬小定理。

<u>Theorem</u> (Fermat's Little Theorem)

$\forall$ integer m and prime p such that g.c.d.$(m,p)=1$, we have $m^{p-1} \equiv 1 \pmod p$.

<u>Proof</u>. $P_G(z_1,z_2,\cdots,z_p) = z_1^p + (p-1)z_p^1$



#(Equivalence classes) $= P_G(m,m,\cdots,m)/|G|$

$= m^p + (p-1)m/p$. Hence,

(p个珠子，m个颜色，只考慮旋轉)

$p \mid m^p + (p-1)m$, $(p,m)=1$

$\Rightarrow p \mid m^{p-1} + (p-1)$. $\Rightarrow \underline{m^{p-1} \equiv 1 \pmod p}$

Definition

A graph is an ordered pair (V, E) such that V is a non-empty set and E is a collection of non-empty subsets of V. V is called the vertex set of G $(V(G))$ and E the edge set of G $(E(G))$. $(|V|$: order, $|E|$: size$)$

(•) If V is finite, then G is a finite graph and G is infinite otherwise.

(•) If E is a collection of non-repeated 2-subsets of V, then G is called a <u>simple graph</u>. Clearly, in this case $|E| \leq \binom{|V|}{2}$. G is a complete graph (simple) if $|E| = \binom{|V|}{2}$, i.e., any two 2-subset of V is included, denoted by $K_{|V|}$.

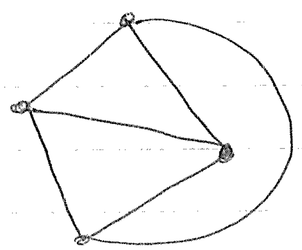
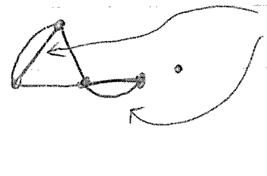


Figure 1, K_4

For convenience, we can draw the graph as in Figure 1.

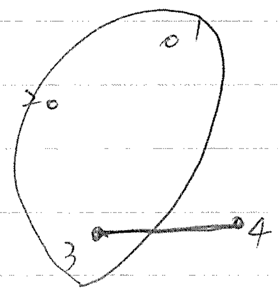
(•) G is a multigraph if E contains repeated 2-subsets, for eample,



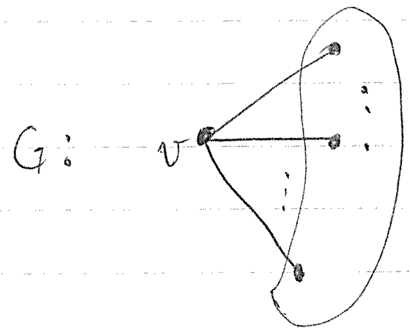
repeated edges

(°) G is a <u>pseudograph</u> if $E(G)$ contains repeated 2-subsets and also 1-subsets (loops).

(°) G is a <u>hypergraph</u> if G is a pseudograph and $E(G)$ contains a subset of $V(G)$ larger than (possibly) 2, e.g., $E = \{\{1,2,3\},\{3,4\}\}$ where $V = \{1,2,3,4\}$.



(*) If we don't mention the kinds of graphs, we refer to a <u>simple graph</u>.



$$N_G(v) = \{u \in V(G) \mid \underbrace{\{u,v\}}_{\text{in short } uv} \in E(G)\}$$

(°) In G, the degree of v is equal to $|N_G(v)|$, denoted by $\deg_G(v)$.
We use $\Delta(G)$ and $\delta(G)$ to denote the maximum and minimum degree resp.

<u>Theorem 1</u> The sum of all degrees in a graph G is equal to $2|E(G)|$. If G is a (p,q)-graph, i.e., G has p vertices and q edges, then $\sum_{v \in V(G)} d_G(v) = 2q$. <u>Proof.</u> One edge contributes "2" degrees. $\blacksquare$

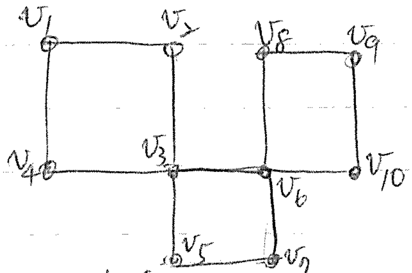
<u>Corollary 1</u> The number of vertices with odd degrees is <u>even</u>.

<u>Proof.</u> The sum is even. $\blacksquare$

For Geometry meaning, $\sum_{v\in V(G)} \deg_G(v)$ is known as the volume of the graph, denoted by $Vol(G) = 2|E(G)| = 2q$.

(**)
Corollary 1 is easy to see, but it has many applications.

Definition (Walk, Trail, Circuit, Path and Cycle)



e.g. $\langle v_1, v_2, v_3, v_6, v_7, v_5, v_3, v_2\rangle$

A walk of a graph G is a sequence of vertices $\langle v_1, v_2, \cdots, v_n\rangle$ in G such that $\{v_i, v_{i+1}\} \in E(G)$.

(∘) A trail of G is a walk of G without repeated edges.

(∘) A circuit of G is a closed trail, i.e., $v_1 = v_n$.

(∘) A path of G is trail without repeated vertices.

(∘) A cycle of G is circuit without repeated vertices.

Definition (Distance of two vertices)

$dist.(u,v) =_{def}$ the number of edges in a shortest path from u to v.

(∘) If there exists no path between u and v, then $\text{dist.}(u,v)=0$.

<u>Definition</u> (Connected graph)

G is a connected graph if for any two vertices in G, there exists a path connecting u and v. (Path-connected)

<u>Definition</u> (Tree)

A connected graph without a cycle (acyclic) is called a tree.

<u>Theorem</u> If T is a tree of order p; i.e., $|V(T)|=p$, then $|E(T)|=\|T\|=p-1$.

<u>Proof.</u> First, we claim that T has at least two vertices of degree 1. Let u and v be the pair of vertices with maximum distance in G (diameter). So, let $\langle u=v_0, v_1, \dots, v_k=v\rangle$ be the path from u to v. Assume that $\deg(u)\geq 2$, then u is adjacent to v_1 and one other vertex, say w. If w is on the path, then we have a cycle. So, w is not on the path. Then, $\text{dist.}(w,v) > \text{dist.}(u,v)$, $\rightarrow\leftarrow$. Hence $\deg(u)=1$. Similary, $\deg(v)=1$.

Now, we can prove the theorem by induction on $|V(G)| = p$.

Clearly, if $p = 1$ or 2, the assertion is true. Let the assertion be true for $p = k$ and consider a tree T of order $k+1$. Since T contains a vertex v of degree 1, $T - v$ is also tree of order k which has $k-1$ edges. This implies that T has k edges. ■

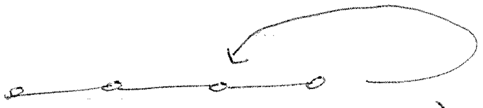
(•) Tree是边最少的連通圖.

(•) Tree中任两点都决定唯一的路径(path).

(*) Tree是最"Beautiful"的圖.
(Useful)

另一个証明Tree有一个Degree"1"的点
(Another proof)

Fact: If every vertex of a graph G ($|G| \ge 3$) of order at least 3 is at least two, then G contains a cycle. ($\delta(G) \ge 2$)

Proof. Suppose not. We may start a walk from a vertex in a component of order ≥ 3. Then,  exists since
(repeated vertex)
$\delta(G) \ge 2$. Then, we have a circuit and thus a cycle. →← ■

Lecture 9 - Continued

Distance in graphs

Let G be a simple graph.

$dist(u,v) =_{def}$ the length of a shortest path connecting u, v.

$dist(u,v) = +\infty$ if u and v are not in the same components.
(or there exist no paths connecting u and v)

distance is a metric defined on graphs

(1) $dist(u,u) = 0$,

(2) $dist(u,v) = dist(v,u)$, and

(3) $dist(u,v) + dist(v,w) \geq dist(u,w)$.

Definitions

(•) Eccentricity of v ($ecc(v)$) : $ecc(v) = \max.\{dist(u,v) \mid u \in V(G)\}$.

(•) diameter of G : $\max.\{ecc(v) \mid v \in V(G)\}$, denoted by $diam(G)$

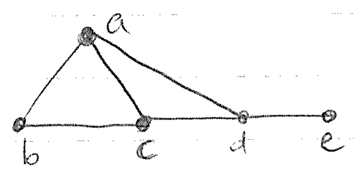
(•) radius of G : $\min.\{ecc(v) \mid v \in V(G)\}$, denoted by $rad(G)$

(•) Center of G : $Z(G) = \{v \mid ecc(v) = rad(G)\}$.

$ecc(a) = 2$
$ecc(b) = 3$
⋮

$diam.(G) = 3$

$rad(G) = 2$

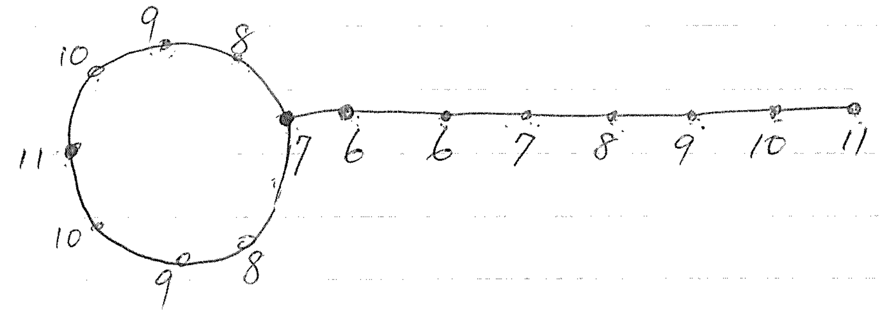


$ecc(c) = 2$, $ecc(d) = 2$

__Theorem__ For any graph G, $rad(G) \le diam(G) \le 2\,rad(G)$.

__Corollary__ For any integers (positive) a and b such that $a \le b \le 2a$, there exists a graph G satisfying $rad(G) = a$ and $diam(G) = b$.

e.g. $a = 6$, $b = 11$.

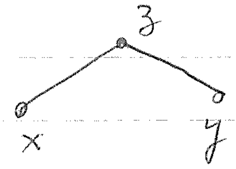


__Proof.__ $rad(G) \le diam(G)$ is trivial.

Let x, y be two vertices in G such that $dist(x,y) = diam(G)$.

Let $z \in Z(G)$, i.e., $ecc(z) = rad(G)$. Now,

$$dist(z,x) + dist(z,y) \ge \underbrace{dist(x,y)}_{= diam(G)}.$$



Since $dist(z,x) \le rad(G)$ and $dist(z,y) \le rad(G)$, we conclude $2\,rad(G) \ge diam(G)$. ∎

(*) __Wiener index__ of G, $W(G) = \sum_{x,y \in V(G)} dist(x,y)$.

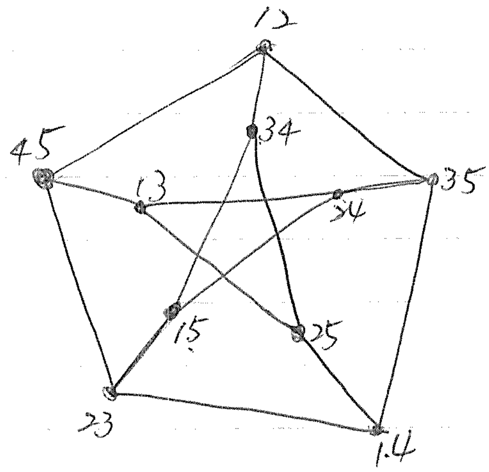
$\Uparrow$

Has many applications!

(∘∘) A graph is called a diameter-k graph if for any two vertices x and y, $dist(x,y) \leq k$ or equivalently $diam(G) \leq k$.

(∘∘∘) The well-known classes in Graph Theory is the so-called diameter-2 graphs.

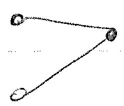
P :



Petersen Graph

(*) The Petersen graph can be labelled (vertices) by using $\binom{\mathbb{Z}_5}{2}$ (two elements subsets) such that two vertices are adjacent if and only if they are disjoint! ($\mathbb{Z}_5 = \{1,2,3,4,5\}$).

(**) Then, $diam(P) = 2$.



Any two non-adjacent vertices do have a common element, say 13 and 23. Hence, they are adjacent to 45.

(*) G is an even (odd) graph if all vertices are of even (odd) degree.

Euler's Circuit Theorem

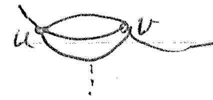
If G is an even connected graph (multigraph), then G contains a circuit passing through all edges of G.

Proof. By induction on the number of edges.

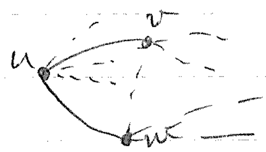
Clearly, it is true for the case that $|E(G)| = 2$. 

Assume that the assertion is true for the case $2 \le |E(G)| \le k$.

Let G be an even connected graph (multigraph) of size $k+1$.

Case 1. G contains two vertices u and v such that uv occurs more than three times, 

. Delete two of them and apply induction.

Case 2. 

The graph G contains at least three vertices u, v, w. Delete uv and uw, add vw, called $\tilde{e}$. Now, apply induction. ∎

Königsberg's Seven Bridges Problem