Mid-Term Discrete Mathematics 2018, 4, 20

1. Use combinatorial argument to verify the following identities.

$$(a)\quad \sum_{k=0}^{m}(-1)^k\cdot\binom{m}{k}\cdot(m-k)^n=0,\quad \text{if}\quad m>n.\quad (10\text{ points})$$

$$(b)\quad \sum_{i=0}^{n}(-1)^i\cdot\binom{n}{i}\cdot\binom{m+n-i}{k-i}=0,\quad \text{if}\quad m<k.\quad (10\text{ points})$$

2. Find the number of solutions $(x_1, x_2, x_3, x_4, x_5, x_6)$ such that all x_i's are positive integers and $x_1 + x_2 + x_3 + x_4 + x_5 + 30x_6 = 101$. (10 points)

3. Find the number of different ways in selecting 13 cards from a regular 52 poker cards such that only two kinds of cards are selected. (10 points)

4. Let $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ be a set of elements. Find the number of permutations α such that $\alpha(x_i) \neq x_i$ for $i \in \{1, 2, 3, 4, 5, 6\}$. (10 points)

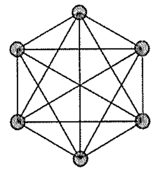
5. Let $a_n = 9a_{n-1} - 15a_{n-2} + 7a_{n-3}$, $a_0 = 0$, $a_1 = 1$ and $a_2 = 2$. Solve the recurrence relation. (10 points)

6. Let $< x_1, x_2, x_3, \ldots, x_{2n} >$ be a $(-1, 1)$-sequence such that both -1 and 1 occur n times. The sequence is called acceptable if the sum of the first k terms is non-negative for each $k = 1, 2, \ldots, 2n$. Find the number of different acceptable sequences when $n = 100$. (15 points) (Explain your answer.)

7. Each edge of the following graph is colored either blue or red. Prove the followings:

   (a) There exists a mono-chromatic triangle, that is a triangle with one color on its edges. (10 points)

   (b) There are two different mono-chromatic triangles. (10 points)



8. Find the determinant of the following matrix. (15 points) (Explain your answer.)

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

9. Prove one of the following two statements for 10 points. (Bonus)

(a) Let ϕ be the Euler function, i.e., $\phi(n)$ is the number of integers k with $1 \leq k \leq n$ such that g.c.d.$(n, k) = 1$. Then, for all $n > 2$, $\phi(n)$ is an even integer.

(b) Let μ be the Möbius function. Then, for all integers $n \geq 2$, $\sum_{d|n} \mu(d) = 0$.

Let $m = p_1^{e_1} \cdot p_2^{e_2} \cdot \cdots \cdot p_s^{e_s}$.

Then, $\mu(m) = \begin{cases} 1, & \text{if} \quad m = 1 \\ 0, & \text{if} \quad e_i \geq 2 \quad \text{for some} \quad i \in \{1, 2, \cdots, s\} \\ (-1)^s, & \text{if} \quad m = p_1 \cdot p_2 \cdot \cdots \cdot p_s \end{cases}$.

離散數學期中考參考解答

1. $m!\left\{ {n \atop m} \right\} = \sum_{k=0}^{m}(-1)^k\binom{m}{k}(m-k)^n = T(n,m)$. ——(1)

(a) 当 $m > n$ 時無法把一个 n 個元素的集合 A 映成(onto)到一个有 m 个元素的集合 B，因此 $T(m,n)=0$. 推導(1)式的过程可参考 Lecture 4, 4-9.

$$\underset{n\text{-set}}{A} \xrightarrow{\text{onto}} \underset{m\text{-set}}{B}$$

(b) Consider $Z = X \cup Y$, $X=\{x_1,x_2,\cdots,x_n\}$ (blue balls)
$Y=\{y_1,y_2,\cdots,y_m\}$ (red balls)

Find the number of k balls selected from Z which contain red balls only. Let E_i be the event, $|E_i|=k$ and $x_i \in E_i$. 因此，利用P.I.E. 不含任何 X 中元素的總數為

$$\binom{m+n}{k} - \underbrace{\binom{n}{1}\binom{m+n-1}{k-1}}_{\text{一个 } x_i} + \cdots + \underbrace{(-1)^n\binom{n}{n}\cdot\binom{m}{k-n}}_{n\text{个 } x_i\text{'s}}$$

$$= \sum_{i=0}^{n}(-1)\binom{m}{i}\binom{m+n-i}{k-i}.$$

如果 $m < k$，則不可能 k 个都只有红色球。($m \geq k$，則答案為 $\binom{m}{k}$.)

2. $(x+x^2+\cdots)^5\cdot(x^{30}+x^{60}+x^{90})$ 求 x^{101} 的係數

$$\Downarrow$$

$$\left(\frac{1}{1-x}-1\right)^5\cdot(x^{30}+x^{60}+x^{90})$$

$$= x^{30}\cdot\left(\frac{x}{1-x}\right)^5 + x^{60}\cdot\left(\frac{x}{1-x}\right)^5 + x^{90}\cdot\left(\frac{x}{1-x}\right)^5 \downarrow$$

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$$= x^{35}\cdot(1-x)^{-5} + x^{65}\cdot(1-x)^{-5} + x^{95}\cdot(1-x)^{-5}$$

$$= (x^{35}+x^{65}+x^{95})\cdot\sum_{k=0}^{\infty}\binom{-5}{k}(-1)^k\cdot x^k$$

$$= (x^{35}+x^{65}+x^{95})\cdot\sum_{k=0}^{\infty}\binom{k+4}{4}x^k$$

x^{101} 的係数为 $\binom{70}{4}+\binom{40}{4}+\binom{10}{4}$。 ▨

3. $$\binom{4}{2}\cdot\left[\binom{13}{1}\cdot\binom{13}{12}+\binom{13}{2}\binom{13}{11}+\cdots+\binom{13}{12}\cdot\binom{13}{1}\right]$$

$$= \binom{4}{2}\left[\binom{26}{13}-12\right] \Rightarrow \binom{4}{2}\left[\binom{13}{0}\binom{13}{13}+\binom{13}{13}\binom{13}{0}\right]$$ ▨

4. The answer is D_6, see Lecture 3, 3-4.

$$D_6 = 6!\cdot\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right)$$

$$= 6\times5\times4\times3 - 6\times5\times4 + 6\times5 - 6 + 1$$

$$= 360-120+30-6+1 = 265.$$ ▨

5. You can use G.F. to solve it, or characteristic polynomial.

See next page

5. $\alpha^3 - 9\alpha^2 + 15\alpha - 7 = 0$

$\alpha^3 - \alpha^2 - 8\alpha^2 + 8\alpha + 7\alpha - 7 = 0$

$\alpha^2(\alpha-1) - 8\alpha(\alpha-1) + 7(\alpha-1) = 0$

$(\alpha-1)(\alpha^2 - 8\alpha + 7) = 0$

$(\alpha-1)^2(\alpha-7) = 0$

$\underline{a_n = c_1 7^n + c_2 + c_3 n}$

6. Catalan number, see Lecture 6, 6-9.

7. Pigeonhole principle (See page 4 for details.)

8. $a_n = a_{n-1} - a_{n-2}$, $\alpha^2 - \alpha + 1 = 0$, ..., See Lecture 6, 6-1, ②

$\Downarrow$

9. Proofs. See Lecture 3, 3-8, 3-9.

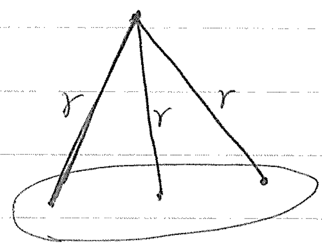
$a_7 = \cos\frac{7\pi}{3} + \frac{1}{\sqrt{3}}\sin\frac{7\pi}{3}$

$= \cos\frac{\pi}{3} + \frac{1}{\sqrt{3}}\sin\frac{\pi}{3}$

$= \frac{1}{2} + \frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{2} = 1.$

7.

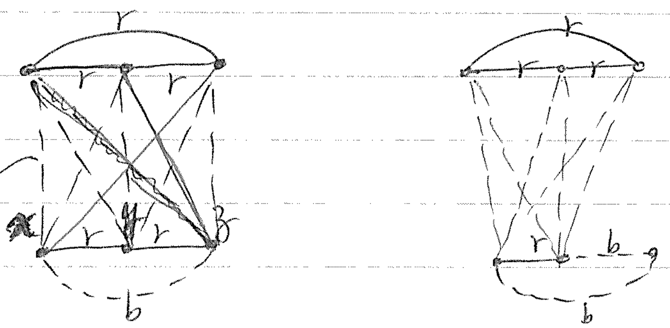
(a)



(One triangle case)

← Either red-triangle or blue triangle

(b)



⇓

Either red or blue

不能有 4个红边，否则形成另一个 r triangle

因此至少有 6个 blue 边.

而且其它各为红边，有红色三角形.