Lecture 5 Special Squares – Continued

One of the most useful Latin squares in applications is called a Latin square with hole.

Definition

Let $\mathbb{Z}_n = \dot{\bigcup}_{i=1}^{k} H_i$. We say: $L = [l_{i,j}]$ is Latin square with hole of type $\prod_{i=1}^{k} h_i$ where $h_i = |H_i|$, provided the following two conditions hold:

(1) $\forall\ l \in \{1, 2, \cdots, k\}$ and $i, j \in H_l$, $l_{i,j}$ is empty; and

(2) $\forall\ x \in \mathbb{Z}_n$, x occurs in each row and each column at most once, furthermore, if $t \in H_l$, then t can not occur in the t-th row and t-column.

	1	2	3	4	5	6	7	8
1			8	5	4	7	6	3
2			6	7	8	3	4	5
3	8	6			7	2	5	1
4	5	7			1	8	2	6
5	4	8	7	1			3	2
6	7	3	2	8			1	4
7	6	4	5	2	3	1		
8	3	5	1	6	2	4		

For convenience, we use $\{1, 2, \cdots, n\} = \mathbb{Z}_n$

(*) A commutative $LS(8)$ with hole of type 2^4.

a commutative $LS(4m+4)$ with hole of type 2^{2m+2}. Therefore, for $2n = 4l$ (when n is even), the square is obtained by embedding of a square of order $2l-2$ into a square $2(2l-2)+4 = 4l$. ■

Reference: Chin-Mei Fu and Hung-Lin Fu, On the intersection of Latin squares with holes, Utilitas Mathematica, Vol. 35, 1989, 67-74.

✓ Bonus (10 points) Construct a commutative $LS(12)$ with holes of type 2^6.

Proposition 3 A Latin square of order km with hole of type k^m exists for each $m \geq 3$.

Proof. Since an idempotent $LS(m)$, M, exists for each $m \geq 3$, the proof follows by the existence of $K \otimes M$ where K is a $LS(k)$. ■

(Here, commutative law is not required.)

(*) If we do a similar square which is commutative, then the proof is harder, but it does exist k and m are larger than 2.

Definition (Complete $LS(n)$)

An $LS(n)$ is called horizontally (row-) complete (resp. vertically (column-)

Proposition 1. For odd n, a commutative $LS(2n)$ with hole of type 2^n exists.

Proof. Let M be an idempotent commutative $LS(n)$. Then let $L = \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} \otimes M$. By deleting all 2×2 subsquares along the diagonal, we have the desired square. ■

$$\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 0 & 2 & 1 \\ \hline 2 & 1 & 0 \\ \hline 1 & 0 & 2 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|}\hline 00 & 01 & 20 & 21 & 10 & 11 \\ \hline 01 & 00 & 21 & 20 & 11 & 10 \\ \hline 20 & 21 & 10 & 11 & 00 & 01 \\ \hline 21 & 20 & 11 & 10 & 01 & 00 \\ \hline 10 & 11 & 00 & 01 & 20 & 21 \\ \hline 11 & 10 & 01 & 00 & 21 & 20 \\ \hline \end{array}$$

$00 \to 0$
$01 \to 1$
$10 \to 2$
$11 \to 3$
$20 \to 4$
$21 \to 5$

Theorem 2. For all $n \geq 3$, a commutative $LS(2n)$ with hole of type 2^n exists.

Proof. It suffices to prove the case when n is even. (Proposition 1 is true.) Since the details are very tedious, we sketch the proof. First, we construct small orders for $n = 4, 6$. Then, we embed a commutative $LS(2m)$ with hole of Type 2^m in

Definition (Diagonal L.S.).

A Latin square $L = [l_{ij}]$ is called a diagonal Latin square, if $\{l_{i,i} \mid 1 \le i \le n\} = \mathbb{Z}_n = \{l_{i,n-i+1} \mid 1 \le i \le n\}$.

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

A diagonal Latin square of order 4

Proposition 6. For each $n \ge 4$, there exists a diagonal Latin square of order n.

Problem. Can you construct such squares?

Theorem 7. (Du et. al) For each $n \ge 4$ and $n \ne 6$, there exists a pair of orthogonal Latin squares of order n.

Corollary 8. For each $n \ge 3$, there exists a magic square of order n.

4	3	8
9	5	1
2	7	6

Proof. $n=3$ can be obtained by direct construction.

For $n \geq 4$, let L and M be two orthogonal diagonal Latin squares of order n. Now, let Q be obtained by superimposing L and M, i.e., $Q = [q_{i,j}]$ where $q_{i,j} = (l_{i,j}, m_{i,j})$.

If we replace $q_{i,j}$ by $n \cdot l_{i,j} + m_{i,j} + 1$, then Q is a magic square.

For example,

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

$\perp$

0	1	2	3
3	2	1	0
1	0	3	2
2	3	0	1

The sum is equal to $n \cdot (0+1+\cdots+n-1) + (0+1+\cdots+n-1) + n$

$$= n \cdot n \cdot \frac{n-1}{2} + \frac{n-1}{2} \cdot n + n$$

$$= \frac{n(n-1)(n+1)}{2} + n$$

$$= \frac{n(n^2+1)}{2}.$$

1	6	11	16
12	15	2	5
14	9	8	3
7	4	13	10

$n = 4$
(Sum = 34)

This is known as a magic square of order 4.

(*) This solution also answers a long-stand problem of finding a systematic construction for magic squares.

complete) if for any ordered pair (α,β) in $\mathbb{Z}_n^2 \setminus \{(i,i) \mid i \in \mathbb{Z}_n\}$, there exists a row (resp. column) of the square in which α and β occur next to each other following the order of α and β.

0	1	2	3
1	3	0	2
2	0	3	1
3	2	1	0

A complete Latin square of order 4.

Furthermore, if L is both horizontally and vertically complete, then L is a complete Latin square.

In general, constructing a complete Latin square is not easy at all. But, if we only consider row-complete or column-complete (not necessarily be both), then it is easier.

__Proposition 4.__ If n is even, then there exists a row-complete (column) Latin square of order n.

__Proof.__ Let $n=2m$. The proof follows by constructing the first row (or column) and then adding $k \pmod n$ to the $k+1^{th}$ row (or column

Now, the first row (or column) is given by

$(0, 2m-1, 1, 2m-2, 2, 2m-3, \cdots, m-1, m)$. So, we have (or its transpose) the desired square since all the differences are distinct (from 1 to $2m-1$). ▨

Example, $m=4$

A complete $LS(8)$ can be obtained by permuting its rows.

$$\begin{Bmatrix} 0 & 7 & 1 & 6 & 2 & 5 & 3 & 4 \\ 1 & 0 & 2 & 7 & 3 & 6 & 4 & 5 \\ 2 & 1 & 3 & 0 & 4 & 7 & 5 & 6 \\ 3 & 2 & 4 & 1 & 5 & 0 & 6 & 7 \\ 4 & 3 & 5 & 2 & 6 & 1 & 7 & 0 \\ 5 & 4 & 6 & 3 & 7 & 2 & 0 & 1 \\ 6 & 5 & 7 & 4 & 0 & 3 & 1 & 2 \\ 7 & 6 & 0 & 5 & 1 & 4 & 2 & 3 \end{Bmatrix}$$

← row complete (column complete can be obtained by finding its transpose)

The following idea was obtained by B. Gordon in 1961.

Theorem 5. If there exists a finite group of order n, $G = \{a_1, a_2, \cdots, a_n\}$ such that $a_1, a_1a_2, a_1a_2a_3, \cdots, a_1a_2\cdots a_n$ are n distinct elements in G, then there exists a complete Latin square of order n.

Proof. Let $b_i = a_1a_2\cdots a_i$, $i = 1, 2, \cdots, n$. Now, construct an $LS(n)$, $L = [l_{ij}]$ where $l_{ij} = b_i^{-1}b_j$. Now, we can check the following properties of L.

(1) Since G is a group, L is indeed a Latin square of order n. (?)

(2) L is row-complete.

It suffices to show that for any two distinct elements α and β in G, there exist s and t such that $\alpha = b_s^{-1} b_t$ and $\beta = b_s^{-1} b_{t+1}$. We claim that s and t can be found uniquely. Since $\alpha = b_s^{-1} b_t$ and $\beta = b_s^{-1} b_{t+1}$, $\beta = \alpha \cdot a_{t+1}$. So, clearly, $a_{t+1} = \alpha^{-1} \cdot \beta$ which gives ~~as~~ unique t for the solution. Now, by $\alpha = b_s^{-1} b_t$, s is uniquely determine since t is fixed.

(3) L is column-complete.

Similarly, we try to find s and t such that $b_s^{-1} b_t = \alpha$ and $b_{s+1}^{-1} b_t = \beta$. Hence $\alpha^{-1} = b_t^{-1} b_s$ and $\beta^{-1} = b_t^{-1} b_{s+1}$, thus $\beta^{-1} = \alpha^{-1} \cdot a_{s+1}$. Now, the proof follows as the argument above. ∎

(*) <u>Exercise 1.9.</u> (Bonus) Construct a complete Latin square of even order by using Theorem 5.