Combin. Des. <u>Lecture 6</u> April 10, 12

Extremal Set Theory

Under a constraint or a collection of constraints find the maximum number of sets satisfying the given constraint(s).

Notations

1. $[n] = \{1, 2, \cdots, n\}$.

2. $\binom{[n]}{k} =_{def}$ the collection of k-subsets (all) of $[n]$.

3. $\binom{n}{k} = \left| \binom{[n]}{k} \right|$.

4. $X = \{x_1, x_2, \cdots, x_n\}$ is a set of n elements and "$\leq$" is a partial order defined on X. $\langle X, \leq \rangle$ is called a partial ordered set, Poset in short.

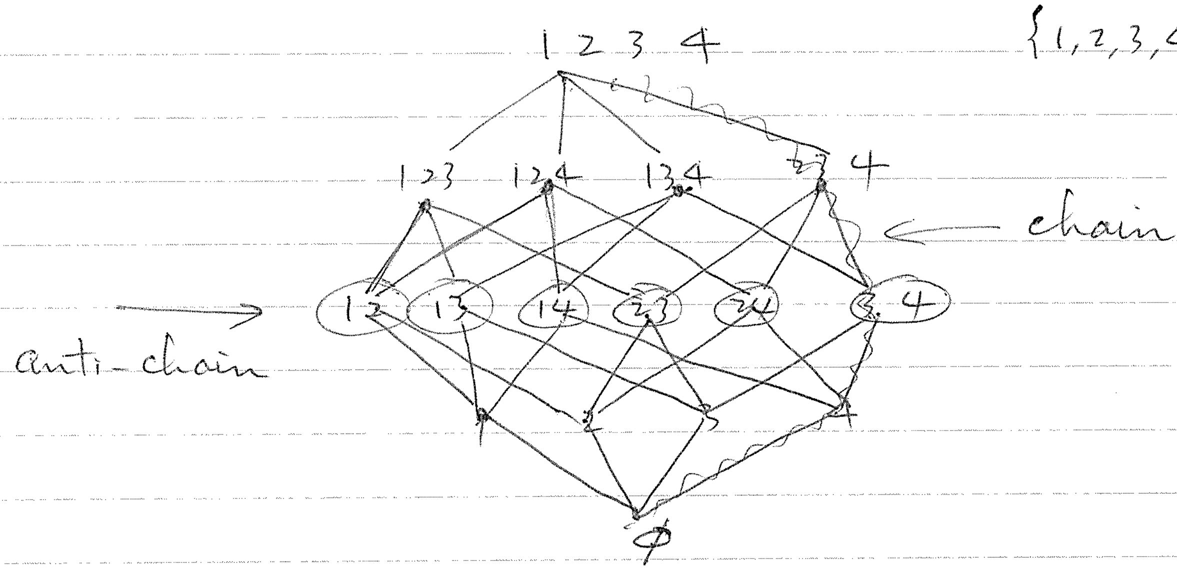
(0) "$\leq$" is a partial order of X if (i) $a \leq a$ $\forall a \in X$, (Reflexivity) (ii) $a \leq b$ and $b \leq a$ implies that $a = b$, $\forall a, b \in X$, (Anti-symmetry) and (iii) $a \leq b$, $b \leq c$, implies that $a \leq c$ $\forall a, b, c \in X$. (Transitivity)

(00) "$\leq$" is a total order of Y provided any two distinct elements in Y, y_i and y_j, either $y_i \leq y_j$ or $y_j \leq y_i$ but not both. (y_i and y_j are comparable.)

(*) A subset of a poset in which no two distinct are comparable is called an anti-chain. On the other hand, a totally ordered set is called a chain.

Example (Poset with set-containment)

$\{1,2,3,4\} \to 1234$



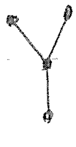
Forbidden poset problem

$P_2 = \mathrm{I}$:

Given a configuration of posets, say  $(y \leq x)$,

find the maximum number of sets in $2^{[n]}$ such that the induced partial ordered set contains no sub-post (I) which is given.

(**) We can change I (" P_2) to $\cdots$. For example, P_3: ,

or  (Y or S_3).
Star

The result solving P_2 case is known as the Sperner's Theorem.

Sperner's Theorem

Consider the collection of all subsets of $[n]$. The maximum number of subsets which do not contain each other is equal to $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. (The maximum anti-chain problem.)

Proof. Let A be the collection of subsets which attains the (a) maximum. Furthermore, let a_k be the number of sets in S whose size is k. Hence, $|A| = \sum_{k=0}^{n} a_k$. (Note that a_i's may be zero.) Since $\binom{[n]}{\lfloor \frac{n}{2} \rfloor}$ is clearly an anti-chain, $|A| \geq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. So, it suffices to prove $|A| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Claim (LYM inequality) $\sum_{k=0}^{n} a_k / \binom{n}{k} \leq 1$.
Lubell-Yamamoto-Mashalkin

Consider the set of permutations of $[n]$. Clearly, there are $n!$. Now, for each set S in A, we associate this set with $|S|!(n-|S|)!$ permutations by taking $\left(\begin{array}{c|c} S & [n] \setminus S \\ \alpha(S) & \alpha([n] \setminus S) \end{array} \right) = \alpha$.

(The first $|S|$ elements of this permutation are exactly S.)

in A ← prefix of a permutation

in A

$\Rightarrow A_1 \subseteq A_2 \rightarrow \leftarrow$

Note that each permutation can only be associated with a single set in A. (?)

$$\sum_{S \in A} |S|!(n-|S|)! = \sum_{k=0}^{n} a_k \cdot k!(n-k)! \leq n!$$

This implies that $\sum_{k=0}^{n} a_k \cdot \frac{k!(n-k)!}{n!} \leq 1$ and the proof follows.

Now, since $1 \geq \sum_{k=0}^{n} a_k \Big/ \binom{n}{k} \geq \sum_{k=0}^{n} a_k \Big/ \binom{n}{\lfloor \frac{n}{2} \rfloor}$, $\binom{n}{\lfloor \frac{n}{2} \rfloor} \geq \sum_{k=0}^{n} a_k = |A|$. ∎

Example $n = 6$

$\{1,2,3\}$	$\{1,3,4\}$
$(1\,2\,3)\,|\,(4\,5\,6)$	$(1\,3\,4)\,|\,(2\,5\,6)$
$(1\,2)(3)\,|\,(4\,5\,6)$	$(1\,4)(3)\,|\,(2\,5\,6)$
$(1\,3)(2)\,|\,(4\,5\,6)$	$(1\,3)(4)\,|\,(2\,5\,6)$
$(1\,3\,2)\,|\,(4\,5\,6)$	$(3\,4)(1)\,|\,(2\,5\,6) \longleftrightarrow \{3,4\}$
$(2\,3)(1)\,|\,(4\,5\,6)$	$(1\,4\,3)\,|\,(2\,5\,6)$
$(1)(3)(2)\,|\,(4\,5\,6)$	$(1)(4)(3)\,|\,(2\,5\,6$
⋮	⋮

Exercise (Bonus) 20 points

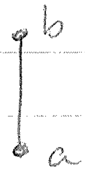
Find the maximum number of subsets in $2^{[n]}$ such that their induced poset does not contain P_3.

A good guess: $\underbrace{\binom{n}{\lfloor \frac{n}{2} \rfloor} + \binom{n}{\lfloor \frac{n}{2} \rfloor + 1}}$. In fact, $ex_p(n; P_3) \geq P(\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1)$.

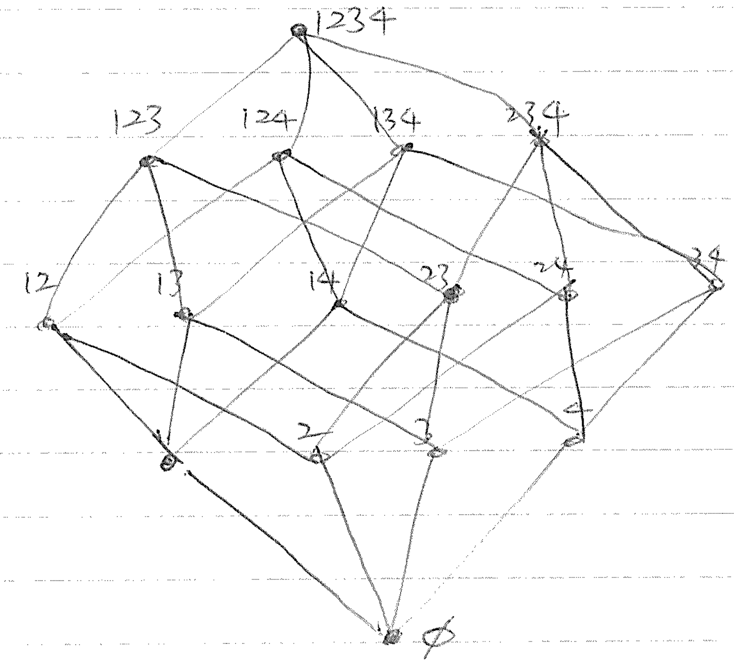
$$\| $$

$$P(\lfloor \tfrac{n}{2} \rfloor, \lfloor \tfrac{n}{2} \rfloor + 1) = \begin{cases} 2\binom{n}{\lfloor \frac{n}{2} \rfloor} & \text{if } n \text{ is odd; and} \\ \binom{n}{\frac{n}{2}} + \binom{n}{\frac{n}{2}+1} & \text{if } n \text{ is even.} \end{cases}$$

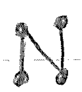
We may use a graph to depict a partial ordered set (Poset), $\langle S, \leq \rangle$. It is known as the Hasse-diagram. Mainly if $a, b \in S$ and $a \leq b$, then the vertex representing b is higher than a as shown in the following:



So, for example, $\langle 2^{\{1,2,3,4\}}, \leq \rangle$ can be represented as follows.



Forbidden Poset Problem

Find the maximum number of subsets (elements) such that its induced subgraph "does not" contain a given graph, for example, $P_2, P_3, \cdots$, , , ...

↑ Not P_4 (in the sense of poset).

Another beautiful result is the maximum collection of sets $B_{n,r}$ of size r which are mutually intersecting, i.e., $\forall\, S_1, S_2 \in B_{n,r}$ (distinct), $S_1 \cap S_2 \neq \phi$. $B_{n,r}$ is called an r-uniform intersecting family defined on $[n]$.

<u>Theorem</u> $|B_{n,r}| = \binom{n-1}{r-1}$ $\forall\, n \in \mathbb{N}$. (EKR Theorem)

<u>Proof.</u> Let $B = \{ S \cup \{n\} \mid S \subseteq \binom{[n-1]}{r-1} \}$. Then, B is an intersecting family of $[n]$ since each set contains the element n. Hence, $|B_{n,r}| \geq \binom{n-1}{r-1}$. Next, we prove that $|B_{n,r}| \leq \binom{n-1}{r-1}$.

Observe that if we let $(a_1, a_2, \cdots, a_n)$ be a cyclic permutation of $[n]$, then this cycle contains at most r sets of $B_{n,r}$. For example, $n = 8$ and $r = 3$, let $(3, 1, 8, 2, 7, 5, 6, 4)$ be an arbitrary cyclic permutation. Now, if $\{8, 2, 7\} \in B_{8,3}$, then we have two more possible sets $\{1, 8, 2\}$ and $\{2, 7, 5\}$. So, for general n, we have at most $r \cdot (n-1)!$ sets in $B_{n,r}$. By the same idea in Sperner's Theorem, each set in $B_{n,r}$ can be associate with $r!(n-r)!$ permutations. So, $|B_{n,r}| \cdot r!(n-r)! \leq r \cdot (n-1)!$, and thus

$$|B_{n,r}| \leq \frac{(n-1)!}{(r-1)!(n-r)!} = \binom{n-1}{r-1}.$$

∎

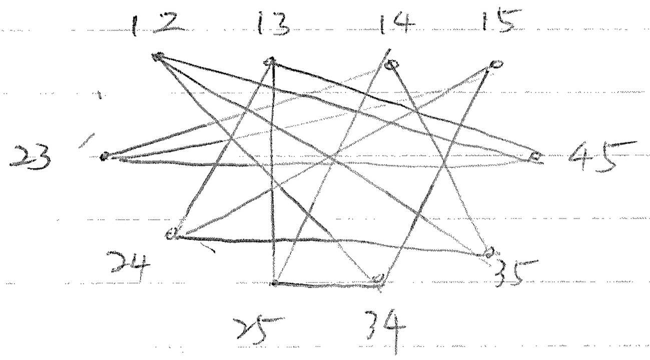
<u>Example</u> $|B_{7,3}| = \binom{6}{2} = 15.$

Another good problem to study related to sets.

Let $n = 2t+1$. Then, we may define a graph G as follows:

$V(G) = \binom{[n]}{t}$ and two vertices are adjacent if and only if their intersection is an empty set.

<u>Example</u> $n=5,\ t=2$



This graph is in fact the Petersen graph.

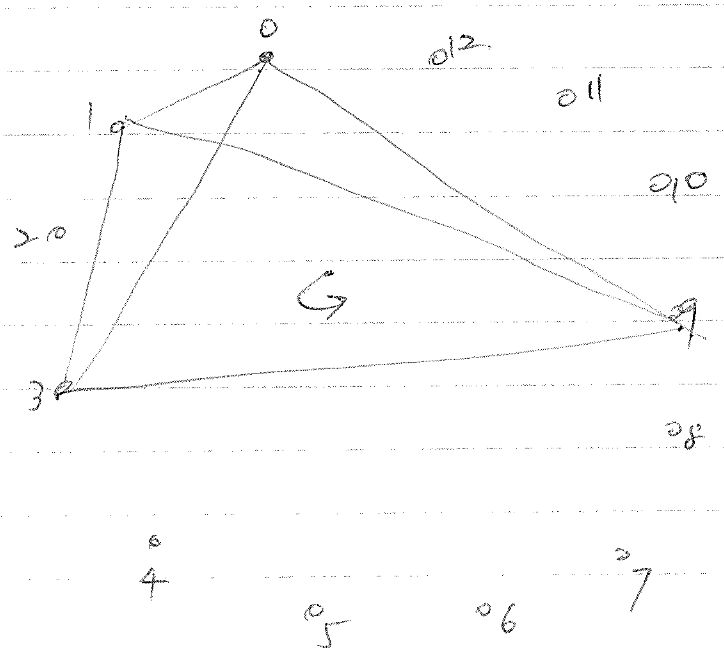
(i) The graph G is known as an order graph of order n, denoted by O_n

(ii) Study the structure of O_n is an important problem in both Graph Theory and Design Theory.

If we further require that any two sets in $B_{n,r}$ can have at most one element in common, thus exactly one element in common, then the collection of such sets in $B_{n,r}$, denoted by $B_{n,r}^{(1)}$ has at most $\frac{n(n-1)}{r\cdot(r-1)}$ sets.

To see this, we notice that any pair of elements in $[n]$ can occur in at most one r-set of $B_{n,r}^{(1)}$. Hence, the pairs we have in total is $n(n-1)/2 = \binom{n}{2}$ and each r-set can use $\binom{r}{2} = \frac{r(r-1)}{2}$ pairs, this implies that $|B_{n,r}^{(1)}| \leq \binom{n}{2} / \binom{r}{2}$.

In fact, for some n and r, the equality does hold. For example, $B_{7,3}^{(1)} = \{124, 235, 346, 457, 561, 672, 713\}$ (Fano plane) and $|B_{13,4}^{(1)}| = \frac{13\cdot 12}{4\cdot 3} = 13$. $B_{13,4}^{(1)} = \{\{0,1,3,9\}+i \mid i \in \mathbb{Z}_{13}\}$.



We can check any two of them have one element in common!