Lecture 5 — March 20, 23

Generating Functions (Continued)

Example 1 Find the number of ways to choose a <u>multiset</u> of cardinality at most 5 from $\{a, b, c\}$ such that at most one b, one c and three a's are selected.

Sol.

$$(1+x+x^2+x^3)(1+x)(1+x) = 1+3x+4x^2+4x^3+3x^4+x^5$$

for a, for b, for c

ϕ	1	2	3	4	5
	$\{a\}$	$\{a,a\}$	$\{a,a,a\}$	$\{a,a,a,b\}$	$\{a,a,a,b,c\}$
	$\{b\}$	$\{a,b\}$	$\{a,a,b\}$	$\{a,a,a,c\}$	
	$\{c\}$	$\{a,c\}$	$\{a,a,c\}$	$\{a,a,b,c\}$	
		$\{b,c\}$	$\{a,b,c\}$		

> You can use
> $(1+ax+a^2x^2+a^3x^3)(1+bx)(1+cx)$
> to figure out the number of ways and also the patterns.

The answer is $1+3+4+4+3+1 = 16$

Example 2 multiset $\longrightarrow$ $\boxed{\text{ordered tuples}}$

e.g. triples (x, y, z)

Sol. $(a,a,a), (a,a,b), (a,b,a), (b,a,a), (a,a,c), (a,c,a), (c,a,a),$
$(a,b,c), (a,c,b), (b,a,c), (b,c,a), (c,a,b), (c,b,a).$

$\left(2\frac{x^2}{1!1!} + 2\frac{x^2}{2!}\right)$

$\left(\frac{x^3}{3!} + 2\frac{x^3}{1!2!} + \frac{x^3}{1!1!1!}\right)$

Use Exponential G.F.

$$\begin{cases} 3!\cdot\frac{1}{3!} & \leftarrow \text{Three } a\text{'s} \\ 3!\cdot\left(2\frac{1}{1!2!}\right) & \leftarrow \text{Two } a\text{'s and } \underline{\text{one } b} \text{ or } \underline{\text{one } c} \\ 3!\,\frac{1}{1!1!1!} & \leftarrow a, b, c \end{cases}$$

<u>有重複元素的排列數</u>

(read n-set-m)

<u>Definition</u> Let $\left\{ {n \atop m} \right\}$ denote the number of ways in partitioning an n-set into m subsets.
non-empty

<u>For example</u>, $\left\{ {3 \atop 2} \right\} = 3$

$\{a, b, c\}$

$$\begin{cases} ab, c \\ a, bc \\ ac, b \end{cases}$$

The # of surjective functions from an n-set onto an m-set is $3\cdot 2!$.

Let $T(n,m)$ denote the number of surjective functions from an n-set onto an m-set. Then

$$(\bullet) \quad T(n,m) = m!\cdot\left\{ {n \atop m} \right\}.$$

所以，我們可以直接求 $\left\{ {n \atop m} \right\}$，也可以先求出 $T(n,m)$ 再除以 $m!$

就是 $\left\{ {n \atop } \right.$

Review

Exponential G.F. for $T(n,m)$.

Let $G(x) = \left(\frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)^m$.

$$G(x) = ((-1) + e^x)^m$$

$$= \sum_{k=0}^{m} \binom{m}{k} (-1)^k \cdot (e^x)^{m-k}$$

$$= \sum_{k=0}^{m} \binom{m}{k} (-1)^k e^{(m-k)x}$$

$$= \sum_{k=0}^{m} \binom{m}{k} (-1)^k \cdot \left(\sum_{j=0}^{\infty} [(m-k)x]^j / j!\right)$$

$$= \sum_{k=0} \binom{m}{k} (-1)^k \cdot \left(\sum_{j=0}^{\infty} (m-k)^j \cdot x^j / j!\right)$$

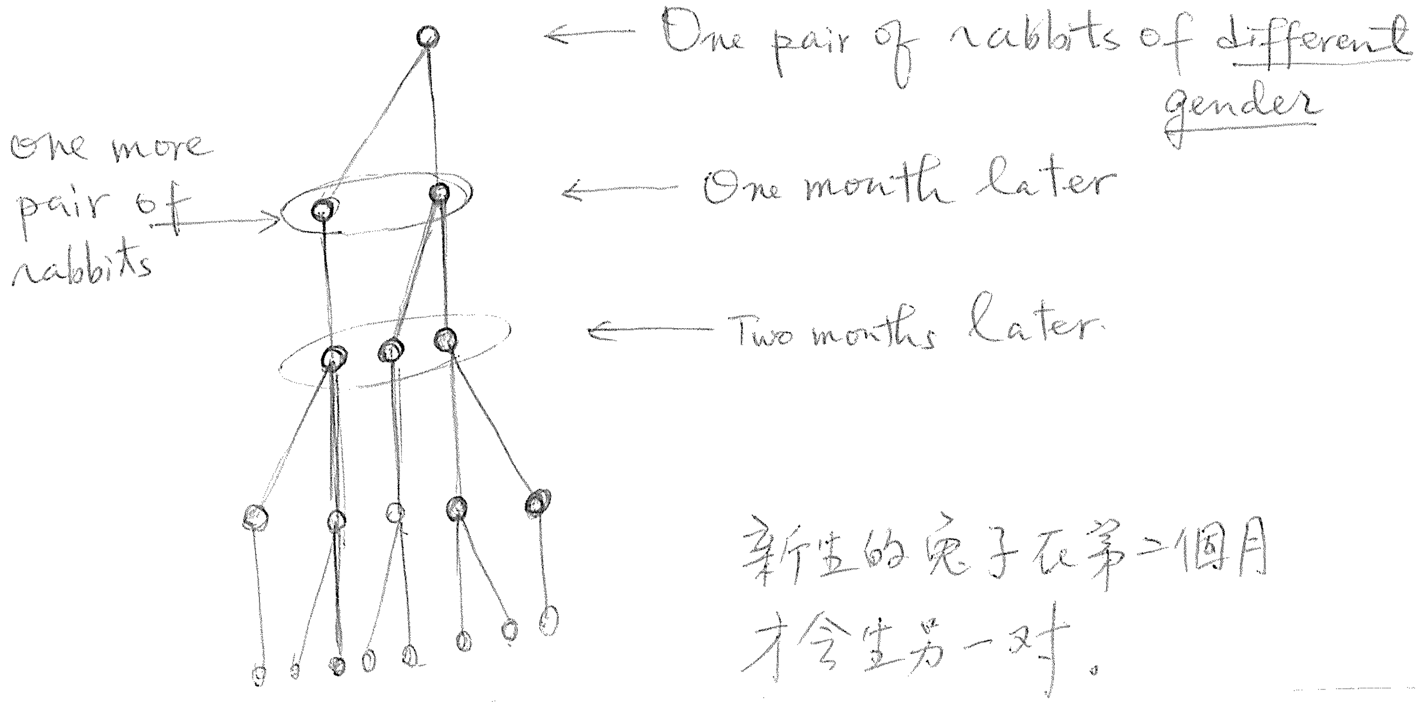
$$= \sum_{j=0}^{\infty} \left(\sum_{k=0} \binom{m}{k} (-1)^k \cdot (m-k)^j\right) x^j / j!$$

$$T(n,m) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k \cdot (m-k)^n$$

$$\left\{ {n \atop m} \right\} = \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^k (m-k)^n.$$

Recurrence Relations (遞迴關係)

(Recursions)



新生的兔子在第二個月
才会生另一对。

How many pairs of rabbits are there in the first day of the n^{th} month?

$F_1 = 1,\ F_2 = 2,\ F_3 = 3,\ F_4 = 5,\ F_5 = 8,\ \cdots,$

$$\underbrace{F_n = F_{n-2} + F_{n-1}.}$$

Recurrence Relation **(Difference Equation)**

About Derrangement

$D_{n+1} = n(D_{n-1} + D_n)$ $\quad \begin{pmatrix} 1 & 2 & 3 & \cdots & n+1 \\ a_1 & a_2 & a_3 & & a_{n+1} \end{pmatrix}$

Clearly, $a_1 \neq 1$.

Let $a_1 = k \neq 1$. Consider a_k.

(1) $a_k = 1$.

$$\begin{pmatrix} 1 & 2 & \cdots & k-1 & k & k+1 & \cdots & n+1 \\ k & \text{———} & & & 1 & \text{———} & & \end{pmatrix}$$

The other part forms D_{n-1}. Since there are n choices for k, we have nD_{n-1}.

(2) $a_k \neq 1$. We may take k as 1, and we have D_n for $\begin{pmatrix} 2 & \cdots & k-1 & 1 & k+1 & \cdots & n+1 \\ a_2 & \text{——} & a_k & & & & \end{pmatrix}$.

Again, there are n choices for a_k, it is nD_n.

也可以直接算，
看下页 5-5'

In total, $D_{n+1} = n(D_n + D_{n-1})$.

<u>Definition</u> (<u>Linear</u> recurrence of rth order).

$$U_{n+r} = a_1 U_{n+r-1} + a_2 U_{n+r-2} + \cdots + a_r U_n.$$

$U_{n+1} = a_1 U_n$, $U_{n+2} = a_1 U_{n+1} + a_2 U_n$, $\cdots$

$$n\cdot(n-1)!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^{n-1}\frac{1}{(n-1)!}\right]$$

$$+\ n\cdot n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^{n}\frac{1}{n!}\right]$$

$$= n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^{n}\frac{1}{n!}\right]-\cancel{n!\cdot(-1)^{n}\frac{1}{n!}}$$

$$+\ n\cdot n!\left[\ \text{———}\ \right]$$

$$= (n+1)!\left[1-\frac{1}{1!}+\cdots+(-1)^{n}\frac{1}{n!}+(-1)^{n+1}\frac{1}{(n+1)!}\right]-(-1)^{n+1}-(-1)^{n}$$

$$-\left[(-1)^{n+1}+(-1)^{n}\right] = 0$$

$$= D_{n+1}$$

(∘) 遞迴關係的獲得，最主要的原因是：

該組合量可以經由前面較小的組合量來組成。

(∘∘) 另一個例子是 $a_n = 2a_{n-1}$，後者為前者的兩倍，

於是當 a_0 決定之後，$a_1 = 2a_0$，$a_2 = 2a_1 = 4a_0$，

$\cdots$，$a_n = 2^n\cdot a_0$。

How to solve a recurrence relation?

An example to start

Let $F_n = F_{n-1} + F_{n-2}$ where $F_1 = 1$, $F_0 = F_{-1} = F_{-2} = 0$.

Let $F(x) = \sum\limits_{k=0} F_k x^k$.

$$\sum_{k=2} F_k x^k = \sum_{k=2} (F_{k-1} + F_{k-2}) x^k = \sum_{k=2} F_{k-1} x^k + \sum_{k=2} F_{k-2} x^k$$

$$= x \sum_{k=2} F_{k-1} x^{k-1} + x^2 \sum_{k=2} F_{k-2} x^{k-2}$$

$$= x(F_1 x + F_2 x^2 + \cdots) + x^2 (F_0 + F_1 x + F_2 x^2 + \cdots)$$

$$(F_2 x^2 + F_3 x^3 + \cdots) = x(F_1 x + F_2 x^2 + \cdots) + x^2 (F_0 + F_1 x + F_2 x^2)$$

$$F(x) - F_0 - F_1 x = x(F(x) - F_0) + x^2 \cdot F(x) \qquad (*)$$

$$F(x) - 0 - x = x(F(x) - 0) + x^2 F(x) = x^2 F(x) + x F(x)$$

$$F(x)[1 - x - x^2] = x, \quad F(x) = \frac{x}{1 - x - x^2}$$

Let $1 - x - x^2 = (1 - \alpha_1 x)(1 - \alpha_2 x) = 1 - (\alpha_1 + \alpha_2) x + \alpha_1 \alpha_2 x^2$.

$$\alpha_1 + \alpha_2 = 1, \quad \alpha_1 \alpha_2 = -1$$

$$\alpha_1 (1 - \alpha_1) = -1 \Rightarrow \alpha_1^2 - \alpha_1 - 1 = 0, \quad \alpha_1 = \frac{1 \pm \sqrt{5}}{2}.$$

Choose $\alpha_1 = \frac{1+\sqrt{5}}{2}$ and $\alpha_2 = \frac{1-\sqrt{5}}{2}$.

$$a\left(1-\frac{1-\sqrt5}{2}x\right)+b\left(1-\frac{1+\sqrt5}{2}x\right)=x$$

$\Rightarrow$ $a+b=0$, $a+b+(b-a)\sqrt5=-2$.

$\Rightarrow$ $a=\frac{1}{\sqrt5}$, $b=-\frac{1}{\sqrt5}$

$$F(x)=\frac{1}{\sqrt5}\left(\sum_{k=0}\left(\frac{1+\sqrt5}{2}\right)^k x^k-\sum_{k=0}\left(\frac{1-\sqrt5}{2}\right)^k x^k\right)$$

$$\Rightarrow F_n=\frac{1}{\sqrt5}\left(\left(\frac{1+\sqrt5}{2}\right)^n-\left(\frac{1-\sqrt5}{2}\right)^n\right). \quad (n\ge 0).$$

(∘) If we change the initial condition to

$$g_0=2,\ g_1=-1 \text{ and } g_n=g_{n-1}+g_{n-2}\ (n\ge 2).$$

Then, only the coefficient "$\frac{1}{\sqrt5}$" will be <u>changed</u>.

10 min. later

$$g_n=\frac{\sqrt5-2}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^n+\frac{\sqrt5+2}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^n.$$

(∘∘) Since in (*) (last page) the next <u>$G(x)$ will be</u> (with different initial conditions) <u>different from $F(x)=\frac{x}{1-x-x^2}$</u>.

For linear homogeneous recurrence relations, we have a better idea in solving the "generating functions".

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k} \quad \text{where } k \geq 1 \text{ and } a_k \neq 0.$$

<u>Recall</u> Solving homogeneous differential equations with constant coefficients.

For example,

$$y'' - 3y' + 2y = 0 \quad \text{where } y = f(x),\ f(0) = \alpha,\ f'(0) = \beta.$$

The basic solutions are of type e^{cx}.

So,

$$(e^{cx})'' - 3(e^{cx})' + 2e^{cx} = 0$$

$$c^2(e^{cx}) - 3c(e^{cx}) + 2e^{cx} = 0$$

$$\Rightarrow c^2 - 3c + 2 = 0, \quad c = 1 \text{ or } 2.$$

The solution is $f(x) = c_1 e^{x} + c_2 e^{2x}$ where $f(0) = \alpha$ and $f'(0) = \beta$.

$$\Rightarrow c_1 + c_2 = \alpha,\ c_1 + 2c_2 = \beta \Rightarrow c_2 = \beta - \alpha \text{ and } c_1 = 2\alpha - \beta.$$

$$\Rightarrow f(x) = (2\alpha - \beta)e^{x} + (\beta - \alpha)e^{2x}.$$

由於 $c^2 - 3c + 2 = 0$ 有兩個相異根，解的一般式就是

$$f(x) = c_1 e^{x} + c_2 e^{2x}.$$

如果形式改成 $y'' - 2y' + y = 0$，則 $c^2 - 2c + 1 = (c-1)^2 = 0$

(*) xe^x is a solution.

$$(xe^x)'' - 2(xe^x)' + xe^x$$
$$= (e^x + xe^x)' - 2(e^x + xe^x) + xe^x$$
$$= e^x + e^x + xe^x - 2e^x - 2xe^x + xe^x = 0$$

三重根，解為 e^x, xe^x, x^2e^x ? 的組合。

Back to generating functions

We shall use α^n for basic solutions!

$$a_n = \sum_{i=1}^{p} c_i a_{n-i}, \quad n \geq p, \quad c_p \neq 0, \quad \underline{a_0, a_1, \cdots, a_{p-1}} \text{ are known values.}$$

(Initial conditions)

Recurrence relation

Corresponding "characteristic equations".

$$x^p - c_1 x^{p-1} - c_2 x^{p-2} - \cdots - c_p = 0$$

e.g. $F_n = F_{n-1} + F_{n-2}$

$\boxed{\text{Characteristic equation is } x^2 - x - 1 = 0.}$

(•) Only homogeneous type is working!

$$x = \frac{1 \pm \sqrt{5}}{2}$$

Sol. $F_n = c_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n.$

$F_0 = 0 \Rightarrow c_1 + c_2 = 0$

$F_1 = 1 \Rightarrow \frac{c_1(1+\sqrt{5})}{2} + \frac{c_2(1-\sqrt{5})}{2} = 1$

$c_1\sqrt{5} - c_2\sqrt{5} = 2$

$2c_1\sqrt{5} = 2, \quad c_1 = \frac{1}{\sqrt{5}}$

$c_2 = -\frac{1}{\sqrt{5}}$

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$ ([illegible])

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e.g. $a_n = 6a_{n-1} - 9a_{n-2}$, $a_0 = 1, a_1 = 2, n \geq 2.$

$x^2 - 6x + 9 = 0 \Rightarrow x = 3$ (multiple root).

Sol. $a_n = c_1 \cdot 3^n + c_2 \cdot n \cdot 3^n$

$c_1 \cdot 1 + 0 = 1$, $c_1 = 1$ and

$3c_1 + 3c_2 = 2, \quad c_2 = -\frac{1}{3}$

$a_n = 3^n - n \cdot 3^{n-1}.$

Note: We can also use generating function to solve

$$a_n = 6a_{n-1} - 9a_{n-2}. \quad a_0 = 1,\ a_1 = 2.$$

Let $G(x) = \sum_{k=0}^{\infty} a_k x^k.$

$$\underline{\sum_{k=2}^{\infty} a_k x^k} = \sum_{k=2}^{\infty} 6a_{k-1}x^k - \sum_{k=2}^{\infty} 9a_{k-2}x^k$$

$$\left(G(x) - a_0 - a_1 x\right) = 6x\left(G(x) - a_0\right) - 9x^2\left(G(x)\right)$$

$$G(x) - 1 - 2x = (6x)G(x) - 6x - 9x^2 G(x)$$

$$G(x)\left(1 - 6x + 9x^2\right) = 1 - 4x$$

$$G(x) = \frac{1-4x}{1-6x+9x^2} = \frac{1-4x}{(1-3x)^2}$$

$$= (1-4x)\cdot(1-3x)^{-2}$$

$$= (1-4x)\cdot \sum_{k=0}^{\infty} \binom{-2}{k}\cdot(-3)^k x^k$$

$$= \sum_{k=0}^{\infty} \binom{-2}{k}(-3)^k x^k - 4\sum_{k=0}^{\infty}\binom{-2}{k}(-3)^k x^{k+1}$$

$$a_n = \binom{-2}{n}(-3)^n - 4\binom{-2}{n-1}(-3)^{n-1}$$

$$= \frac{(-2)(-3)\cdots(-n-1)}{n!}(-3)^n - 4\left(\frac{(-2)(-3)\cdots(-n)}{(n-1)!}\right)(-3)^{n-1}$$

$$= (n+1)\cdot 3^n - 4n\cdot 3^{n-1}$$

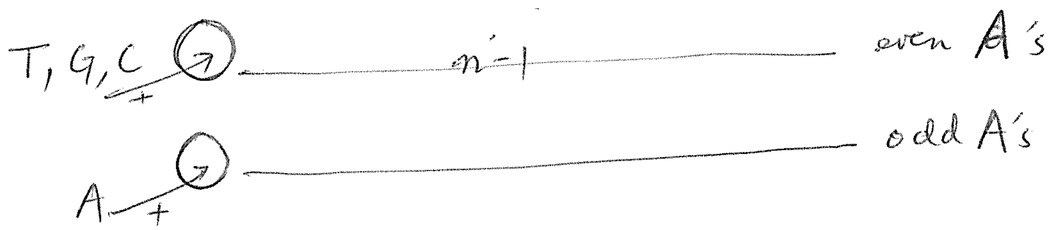
(◎) 用生成函數的方法的確比較複雜，但是，它对於一般不是 Homogeneous 的形式也可以解決问題!

<u>Example</u>

Among 4^n DNA sequence (using A, T, G, C) (n-letter), how many of them have an even number of A's? How many of them have an even number of A and and even number of T's?

(長度為 n 的 DNA 序列)

<u>Sol.</u> Let a_n denote the number of n-letter DNA sequence with an even number of A's. Therefore, a_{n-1} denotes the number of $(n-1)$-letter DNA sequences that have an even number of A's. Hence, the # of $(n-1)$-letter DNA sequences with an odd number of A's is $\underline{4^{n-1}-a_{n-1}}$.



$$a_n = 3a_{n-1} + 4^{n-1} - a_{n-1} = 2a_{n-1} + 4^{n-1}.$$

$$a_n - 2a_{n-1} = 4^{n-1}$$

Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$. $\quad \sum_{k=1}^{\infty} (a_k - 2a_{k-1}) x^k = \sum_{k=1}^{\infty} 4^{k-1} x^k$.

$$\sum_{k=1}^{\infty} a_k x^k - 2x \sum_{k=1}^{\infty} a_{k-1} x^{k-1} = x \sum_{k=1}^{\infty} 4^{k-1} x^{k-1}.$$

$$G(x) - a_0 - 2x\,G(x) = x \cdot \frac{1}{1-4x} \quad , \quad a_0 = 1$$

even # of A's

$$G(x) - 1 - 2x\,G(x) = \frac{x}{1-4x}$$

$$(1-2x)\,G(x) = \frac{x}{1-4x} + 1 \ ; \quad G(x) = \frac{1/2}{1-4x} + \frac{1/2}{1-2x} \quad (5 \text{ min.})$$

$$G(x) = \frac{1}{2}\left(\sum_{k=0}^{\infty} (4x)^k + \sum_{k=0}^{\infty} (2x)^k \right)$$

$$= \frac{1}{2}\left(\sum_{k=0}^{\infty} 4^k x^k + \sum_{k=0}^{\infty} 2^k x^k \right)$$

$$a_n = \frac{1}{2}(4^n + 2^n), \quad n \geq 0.$$

done for part 1

Part 2 is more complicate

a_{n-1} : even A's

b_{n-1} : even A's and even T's ($(n-1)$-letters).

c_{n-1} : even A's and odd T's

d_{n-1} : odd A's and even T's

$e_n = 4^{n-1} - b_{n-1} - c_{n-1} - d_{n-1}$: odd A's and odd T's

(•) We are aiming at finding b_n, so are c_n and d_n.

$n-1$

$$b_n = \overset{\text{加G,C}}{2\,b_{n-1}} + \overset{\text{加T}}{c_{n-1}} + \overset{\text{加A}}{d_{n-1}}$$

$n-1$

$$c_n = \overset{\text{加T}}{b_{n-1}} + \overset{\text{加G,C}}{2\,c_{n-1}} + \overset{\text{加A}}{e_n}$$

$$d_n = b_{n-1} + 2d_{n-1} + e_n.$$

$n-1$

用類似的概念，令

$$B(x)=\sum_{k=0}^{\infty} b_k x^k,\quad C(x)=\sum_{k=0}^{\infty} c_k x^k \text{ and } D(x)=\sum_{k=0}^{\infty} d_k x^k,$$

推得

$$\begin{cases} B(x)-\frac{3}{4}=2x\,B(x)+x\,C(x)+x\,D(x) \\ C(x)-\frac{1}{4}=x\,C(x)-x\,D(x)+\frac{x}{1-4x} \\ D(x)-\frac{1}{4}=-x\,C(x)+x\,D(x)+\frac{x}{1-4x} \end{cases}$$

解联立方程

$$B(x)=\frac{1/4}{1-4x}+\frac{1/2}{1-2x}$$

$$C(x)=D(x)=\frac{1/4}{1-4x}$$

$$\Rightarrow \quad b_n=\frac{1}{4}\cdot 4^n+\frac{1}{2}\cdot 2^n=4^{n-1}+2^{n-1}$$

$$c_n=d_n=4^{n-1}.$$

∎