Lecture 4 March 13, 16.

10 N.T. Dollars : How many ways to combine coins?

1 5 10

x_1 x_2 x_3

$x_1 + 5x_2 + 10x_3 = 10$ Find all integer-solutions for (x_1, x_2, x_3).

We can simply list the answers : (0,0,1), (0,2,0), (5,1,0), (10,0,0).

(*) But, it is not easy to combine coins for 10 U.S. dollars.

(We may assume that we don't have half-dollar and one-dollar coins, in fact, they do exist.)

1 5 10 25

penny nickle dime quarter

So, we have an equation to solve

$$x_1 + 5x_2 + 10x_3 + 25x_4 = 1{,}000,-$$ (Not easy to list the answers!)

(o) By using combinatorial selections, we can consider the product of $(1+x+x^2+\cdots)(1+x^5+x^{10}+x^{15}+\cdots)(1+x^{10}+x^{20}+x^{30}+\cdots)(1+x^{25}+x^{50}+x^{75}+\cdots)$

and the answer is going to be the coefficient of x^{1000}.

Again, this is not an easy problem to solve at all.

Definition (Generating function)

Let $\langle a_0, a_1, a_2, \cdots, a_n, \cdots \rangle$ be a sequence. Then, the function $f(x) = \sum_{i=0}^{+\infty} a_i x^i$ associate with the sequence is called the generating function for the sequence.

It is not difficult to realize that $f(x)$ is in fact a power series with one indeterminate x. If $f(x)$ has an explicit form, then solving a complicate solution may turn to be a simpler one.

Example 1 Let $f(x) = (1+x+\cdots)\cdot(1+x^5+x^{10}+\cdots)\cdot(1+x^{10}+x^{20}+\cdots)$.

Then, $f(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^{10}}$. Let $\frac{1}{1-x} = \sum_{i=1}^{+\infty} A_i x^i$, $\frac{1}{(1-x)(1-x^5)} = \sum_{i=1}^{+\infty} B_i x^i$ and $\sum_{i=1}^{+\infty} C_i x^i = f(x)$. We are aiming at finding C_{100}.

Hence, $\sum_{i=1}^{\infty} A_i x^i = (1-x^5) \cdot \sum_{i=1}^{\infty} B_i x^i \Rightarrow A_n = B_n - B_{n-5}$, i.e.,

$$\begin{cases} B_n = A_n + B_{n-5}, & \text{Similarly} \\ C_n = B_n + C_{n-10}. & A_0 = B_0 = C_0 = 1. \end{cases}$$

Now, $A_n = 1$ for all n. $C_{100} = B_{100} + C_{90} = B_{100} + B_{90} + C_{80} = \cdots$

$= B_{100} + B_{90} + \cdots + B_{10} + C_0 = A_{100} + B_{95} + B_{90} + \cdots + B_{10} + C_0$

$= A_{100} + A_{95} + 2B_{90} + B_{80} + \cdots = A_{100} + A_{95} + 2A_{90} + 2B_{85} + B_{80} + \cdots = 121.$

<u>Example 2</u> Suppose that we have n collection of distinguished items. If we are allowed to choose the items from the same collection, then the number of different combination is denoted by H_r^n provided r items are selected from n items. (可重複組合).

Now, we let $f(x) = (1+x+x^2+\cdots)^n = \left(\frac{1}{1-x}\right)^n$ and H_r^n is equal to the coefficient of x^r in the expansion. For completeness, we provide the details in what follows.

$$f(x) = (1-x)^{-n} = (1+(-x))^{-n} = \sum_k \binom{-n}{k}(-x)^k = \sum_k \binom{-n}{k}(-1)^k x^k.$$

By the definition* of $\binom{-n}{k}$, we have

$$H_r^n = \binom{-n}{r}(-1)^r = \frac{(-n)(-n-1)\cdots(-n-r+1)}{1\cdot 2\cdot\cdots\cdot r}\cdot(-1)^r = \frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r}.$$

<u>Exercise 1</u> Find the explicit form of $f(x) = \sum_{i=0}^{\infty}(i+1)x^i$.

<u>Exercise 2</u> Find the number of partitions of 101 into positive odd integers.

<u>Exercise 3</u>. Find the number of partitions of n into mutually distinct positive integers.

$$(*) \quad \binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!} \quad \text{for each } x\in\mathbb{R}.$$

1. Find the # of non-negative integer solutions of $x_1+x_2+x_3+\cdots+x_m=n$.

Use G.F. Let

$f(x)=(1+x+x^2+\cdots)^m$ and find the coefficient of x^n.

$$f(x) = \left[\frac{1}{(1-x)}\right]^m = (1-x)^{-m}$$

$$= \sum_{k=0} \binom{-m}{k}(-x)^k = \sum_{k=0} \binom{-m}{k} (-1)^k x^k$$

$$= \sum_{k=0} \frac{(-m)(-m-1)\cdots(-m-k+1)(-1)^k}{k!} x^k$$

The coefficient of x^n is equal to

$$\frac{(+m)(m+1)\cdots(m+n-1)}{n!} = \binom{m+n-1}{n}.$$

2. Find the # of positive integer solutions of $x_1+x_2+\cdots+x_m=n$.

Use G.F.

$$g(x) = (x+x^2+\cdots+x^t+\cdots)^m$$

$$= x^m(1+x+\cdots) = x^m\cdot(1-x)^{-m}$$

$$= x^m \sum_{k=0}^{\infty} \binom{m+k-1}{k} x^k$$

$$= \sum_{k=0}^{\infty} \binom{m+k-1}{k} x^{m+k}. \qquad \left[\begin{array}{l}\text{Let } n=m+k. \\ \Rightarrow k=n-m.\end{array}\right]$$

Hence, the coefficient of x^n is $\binom{n-1}{n-m} = \underline{\binom{n-1}{m-1}}$.

(*) Use $x_i' = x_i - 1$, solve $x_1'+x_2'+\cdots+x_m' = n-m$. (Another method!)

4-5

3. Applications on partitions $p(n)$.

① $$\frac{1}{1-x} = 1+x+x^2+\cdots$$
$$= (1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^r})\cdots$$
$$= (1+x+\cdots+x^9)(1+x^{10}+x^{20}+\cdots+x^{90})(1+x^{100}+x^{200}+\cdots+x^{900})\cdots$$

② $(1+x)(1+x^2)(1+x^3)\cdots(1+x^m)\cdots$ 分成不相等的整数和

$$= \frac{1-x^2}{1-x}\cdot\frac{1-x^4}{1-x^2}\,\frac{1-x^6}{1-x^3}\cdot\frac{1-x^8}{1-x^4}\cdots\frac{1-x^{2m}}{1-x^m}\cdots$$

$$= \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}$$

$$= (1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)$$

分成奇数项的和

Example

$7 = $

7	7
$6+1$, $4+2+1$,	$5+1+1$
$5+2$	$3+3+1$
$4+3$	$3+1+1+1+1$
	$1+1+1+1+1+1+1$

$\| \; P_d(7)=5$ $\qquad$ $\| \; P_o(7)=5$

(•) Find some exercises to work on, good luck!

4. An interesting generating function.

$$(1-4x)^{-\frac{1}{2}}$$

$$= \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \cdot (-4x)^k$$

$$= \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{1}{2}-k+1)}{k!} (-4)^k \cdot x^k$$

$$= \sum_{k=0}^{\infty} \frac{2^k \cdot (1\cdot 3\cdot 5 \cdots \cdot (2k-1))}{k!} x^k$$

$$= \sum_{k=0}^{\infty} \frac{2^k \cdot k!\,(1\cdot 3\cdot 5\cdot \cdots \cdot (2k-1))}{k!\cdot k!} x^k = \sum_{k=0}^{\infty} \frac{(2\cdot 4\cdot \cdots 2k)(1\cdot 3\cdot 5\cdots (2k-1))}{k!\,k!}$$

$$= \sum_{k=0}^{\infty} \binom{2k}{k} x^k.$$

(••) We shall use this G.F. later in various counting problems.

Instead of using the power series $\sum_{i=0}^{\infty} a_i x^i$ to represent the generating function for the sequence $\langle a_0, a_1, a_2, \cdots \rangle$, we may use a more general form to describe the generating function.

<u>Definition</u> (<u>General GF</u>) Let $\mu_k(x)$ be a function satisfying $\sum_{k=0}^{\infty} a_k \mu_k(x) \neq \sum_{k=0}^{\infty} b_k \mu_k(x)$ provided $\langle a_0, a_1, \cdots \rangle$ and $\langle b_0, b_1, \cdots \rangle$ are two distinct sequences (non-equal sequences). Then, $F(x) = \sum_{k=0}^{\infty} a_k \mu_k(x)$ is a generating function of the sequence $\langle a_0, a_1, \cdots \rangle$ and $\mu_k(x)$ is called the indicator function for a_k.

Note that in previous definition for generating function, we choose $\mu_k(x) = x^k$. In what follows, we consider $\underline{\mu_k(x) = x^k/k!}$.

<u>Definition</u> (Exponential Generating Functions)

$F(x) = \sum_{k=0}^{\infty} a_k x^k/k!$ is called the exponential generating function of the sequence $\langle a_0, a_1, \cdots \rangle$.

The notion "exponential" comes from the fact that e^x is the exponential generating function of the sequence $\langle 1, 1, \cdots \rangle$. Another important fact shows that we have permutation of items into our counting.

Examples (E.G.F.)

1. Let $P(n,k)$ denote the number of permutations of k objects selected from a set of n objects.

e.g. $n=4$, $k=3$, $S=\{1,2,3,4\}$.

$P(n,k) : \binom{n}{k} \cdot k!$

6 permutations for each 3-subset of S.

$\{123, 213, 321, 132, 231, 312\}$ for $\{1,2,3\}$.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} k! \cdot \binom{n}{k} \frac{x^k}{k!}$$

E.G.F. of $P(n,k)$.

2. In DNA (sequences), we use A, T, G, C four letters to form a sequence. Suppose that we construct a length k sequence by using at most 3 A, 3 T, 1 G and 2 C. How many ways we can have if $k=4$.

By the idea of E.G.F. $g(x) = \left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}\right)^2\left(1+\frac{x}{1!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}\right)$

We are looking for the coefficient of $\frac{x^4}{4!}$.

$$\sum_{k=0}^{\infty} a_k \frac{x^k}{k!}$$

Again, let $k=6$.

If we choose 2A, 2T and 2C, then we shall have

$$\frac{x^2}{2!}\cdot\frac{x^2}{2!}\cdot\frac{x^2}{2!}.$$

$\therefore$ A, A, T, T, C, C form a sequence with $\frac{6!}{2!2!2!}$ ways.

Similarly, 3A, 2T, 1G, we have

$$\frac{x^3}{3!}\cdot\frac{x^2}{2!}\cdot\frac{x}{1!}, \text{ with } \frac{6!}{3!2!1!} \text{ ways}$$

(0) 由於 a_6 為 $x^6/6!$ 的係數，所以
乘開來之後，分子就多了 $6!$，剛好是有相同物件的排列數。

Now, we consider the # of surjective functions from a set $A=\{a_1,a_2,\cdots,a_n\}$ onto a set $B=\{b_1,b_2,\cdots,b_m\}$. Since the image of a function has order, we use exponential G.F.

$G(x)=\left(\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)^m$ and try to find a_n in $\sum_{k=0}^{\infty}a_k(x^k/k!)$.

$$G(x)=(-1+e^x)^m$$

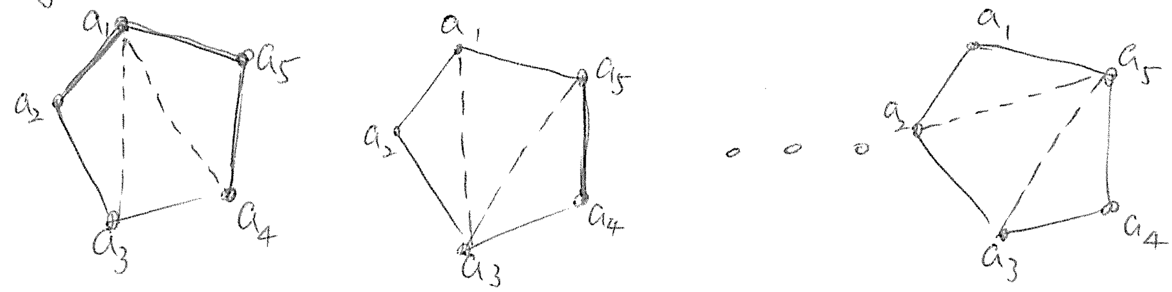
$$=\sum_{k=0}^{m}\binom{m}{k}\cdot(-1)^k\cdot\underbrace{(e^x)^{m-k}}_{e^{(m-i)x}}$$

$$=\sum_{k=0}^{m}\binom{m}{k}(-1)^k\cdot\sum_{j=0}^{\infty}\frac{[(m-k)x]^j}{j!}=\sum_{k=0}^{m}\binom{m}{k}(-1)^k\cdot\left(\sum_{j=0}^{\infty}(m-k)^j\cdot x^j/j!\right)$$

$$=\sum_{j=0}^{\infty}\boxed{\left[\sum_{k=0}^{m}\binom{m}{k}(-1)^k\cdot(m-k)^j\right]}x^j/j! \Rightarrow j=n \text{ 時為答案。}$$

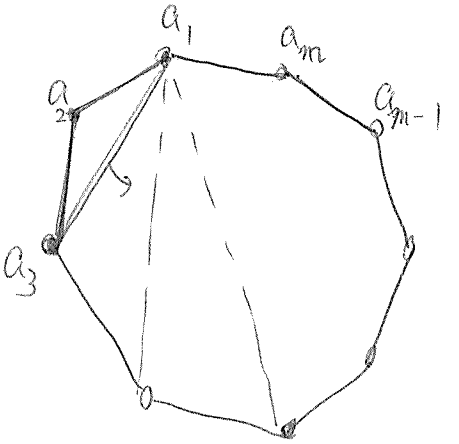
Example Triangulate a convex m-gon.
(How many ways)

e.g. $m = 5$

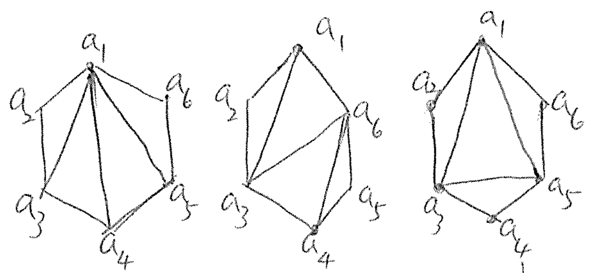


Five ways.

How about m in general?



$m = 6$



There are 11 other ways

5 5 1

more

For convenience, we consider the triangulation of an $(n+2)$-gon. So, how many ways? (Recommended Exercise!)

Let d_n be the number of ways in triangulating an $(n+2)$-gon.

Then, by using G.F., we may let $g(x) = \sum_{k=1}^{\infty} d_k x^k$.

⋮

To be continued next week!