

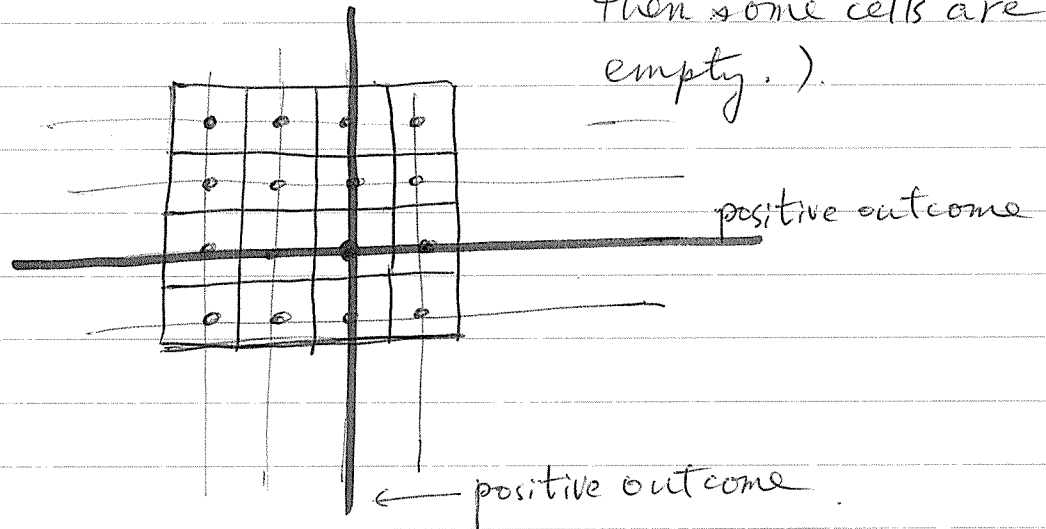
Grid Designs

For n items, choose an $m \times m$ grid where $m \geq \sqrt{n}$.

Put all n items at grid points. Take each grid line as a test. If $d=1$, then the positive item can be identified at the intersection of two lines receiving positive outcome.

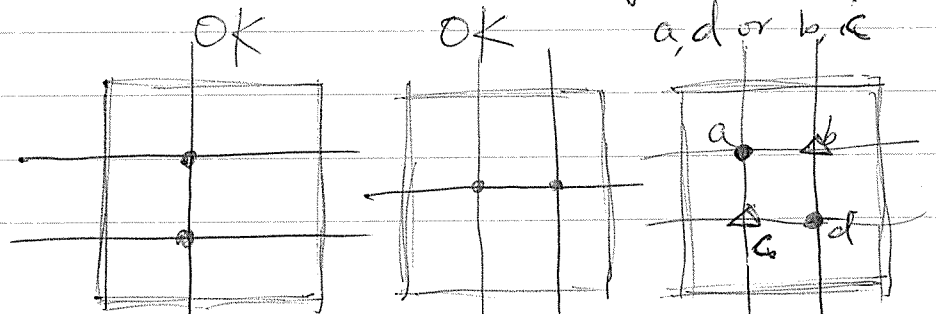
For example, $n=16$,

(If n is not a square, then some cells are empty.)



8 tests of a grid design

As a matter of fact, 8 tests are ~~too~~ many in the sense of optimality. But, we can extend the above design to find "two positives".



We can apply one more individual test to either a or b to decide which pair is positive. Or, we use two grids to solve the case $d=2$.

(C1) For any three items a, b and c, it occurs at most once that a is collinear with b on one line and also collinear with c on another line.

e.g. $n=16$, $d=2$, 8 and 10 are positives.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

6 and 8 } are not on a line
6 and 10 } simultaneously

Note that the condition (C1) is not only for 6+8, 6+10,

any three items should satisfy the condition. So, it is left

to consider how to construct such two grids. The set

of grids we construct is known as a grid-design.

1	2	3	1	5	9
4	5	6	2	6	7
7	8	9	3	4	8

Suspect set $\{5, 6, 8, 9\}, \{6, 4, 7, 8\}$.

$$\{5, 6, 8, 9\} \cap \{4, 6, 7, 8\} = \{6, 8\} \leftarrow \text{Two positives.}$$

Now, consider $d > 2$. For example, $d = 3$.

	ab	a		(ad)
	b	(ab)		(bc)
	(bd)	(bd)		c

$3, 12, 20$ are positives

(*) We need an extra grid (satisfying (C1)) to identify the items ab and $(ab)'$, ac and $(ac)'$, bc and $(bc)'$ respectively.

(**) $\binom{d}{2} + 1$ grids are needed if only (C1) condition is needed for the grid.

(*) The intersection of 4 suspect set ($d=3$) is the set of positives.

(*) In case that for $d=3$, we only obtain (< 6) outcomes which are positive, then the number of grids can be smaller.

But, in general, we have no idea where are the positives.

(*) We use $\left[\binom{d}{2} + 1 \right] \cdot \lceil \sqrt{n} \rceil \cdot 2$ tests in grid design (C1 condition)!

A better idea of design

(C2) Every two items can be collinear in at most one grid.
(Unique Collinearity Condition)

Theorem (F.K. Hwang, JGT(19), 1995, no.3, 333-337).

$\left\lceil \frac{d+1}{2} \right\rceil$ grids satisfying (C2) are enough to identify d positives.

It is worth of noting here that we can use these grids to construct a d -disjunct matrix. This is why this grid design is beautiful.

Proof. Suppose not. Then, there exists a negative item which can not be identified by $\left\lceil \frac{d+1}{2} \right\rceil$ grids. By the explanation earlier, this item must be collinear with at least two positives in each grid. By (C2), these positives are distinct. Hence, in total, we have at least $\left\lceil \frac{d+1}{2} \right\rceil \cdot 2$ positives, a contradiction. \square

$$\binom{\checkmark}{d}$$

Example $d=2$

1	2	3		1	5	9
4	5	6		6	7	2
7	8	9		8	3	4

An example satisfying (C2).

By Theorem (Kwang), two such grids are enough for identifying "two" positives.

(•) For example, if "5" is negative and we are not able to identify "5" since 5 is collinear with positives "6" and "8" as mentioned earlier ~~if~~. But, in the second grid, 5 is not collinear with "6", and "8", by (C2), hence 5 can be identified as "negative".

Another fact

If we take each row and each column of the above two grids as a set, then we have an affine plane of order 3.

$$B = \left\{ \begin{array}{l} \{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\} \\ \{1, 5, 9\}, \{2, 6, 7\}, \{3, 4, 8\}, \{1, 6, 8\}, \{3, 5, 7\}, \{2, 4, 9\} \end{array} \right\}, \quad X = \{1, 2, \dots, 9\}$$

Construction of grids satisfying (C2) when $n = k^2$

0	0	0
1	1	1
2	2	2

0	1	2
0	1	2
0	1	2

0	1	2
2	0	1
1	2	0

0	2	1
2	1	0
1	0	2

orthogonal 3x3 arrays.

⇓

00	01	02
10	11	12
20	21	22

⇓

00	12	21
22	01	10
11	20	02

orthogonal Latin Squares.
(How to construct?)
Next Semester

$(i, j) \rightarrow 3i + j$

0	1	2
3	4	5
6	7	8

0	5	7
8	1	3
4	6	2

Grids

One way to construct. But, in general, the idea, in "6" is much better!
mentioned

Theorem (Hwang)

There exist g $k \times k$ grids satisfying (C2) if and only if there exist $2(g-1)$ mutually Latin squares (2g orthogonal arrays of order k).

Proof. See 6.

Let the set of $2(g-1)$ MOLS(k) be $L_1, L_2, \dots, L_{2(g-1)}$ and the two index-arrays be L_r and L_c . Now, by combining two arrays, L_r and L_c , L_1 and L_2, \dots, L_{2g-3} and L_{2g-2} (See 6'' for reference), we obtain g grids, G_0, G_1, \dots, G_{g-1} . Now, it suffices to show that these grids do satisfy (C2).

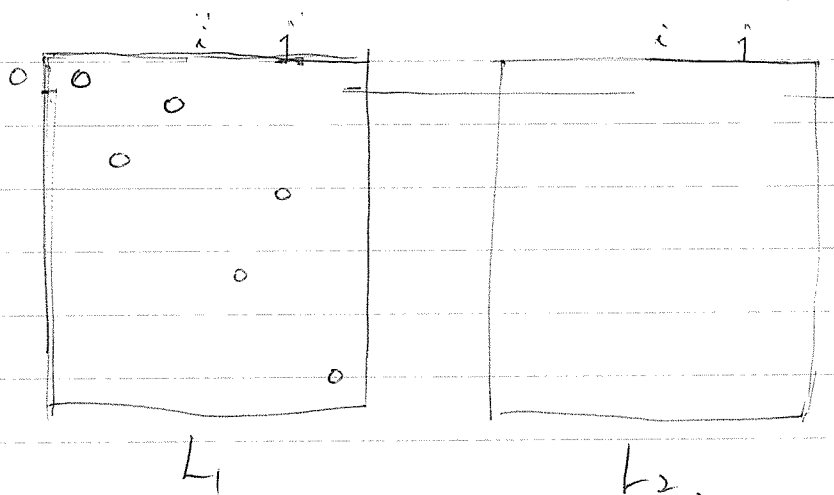
By orthogonality, each element of $\mathbb{Z}_k^2 \setminus \{0, 1, \dots, k^2-1\}$ occurs exactly once in each grid. Now, consider two elements x and y in \mathbb{Z}_k^2 .

Let x and y be collinear in G_0 without loss of generality.

Say, in the first row. Then, they have the same entry "0" in

L_1 . Now, we claim that x and y can not be collinear in

the other grids. Both cases for x and y in the same row or column will lead to a contradiction.



See the idea in 6'' and 6'''.

(*) The idea of combining two Latin squares into one grid (orthogonal)

e.g.

	0	1	2	3	4
0	0	1	2	3	4
1	4	0	1	2	3
2	3	4	0	1	2
3	2	3	4	0	1
4	1	2	3	4	0

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

↓

	0	1	2	3	4
0	00	33	11	44	22
1	23	01	34	12	40
2	41	24	02	30	13
3	14	42	20	03	31
4	32	10	43	21	04

← 左边填 0, 右边分别填 0, 1, 2, 3, 4 的位置

← 右边填 2, ...

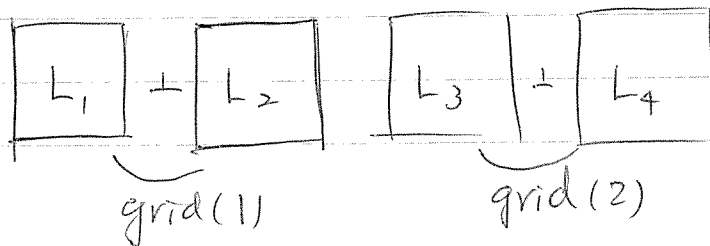
→ 上面 (2, 3) 出现在 (3, 0) 的位置

↑ 右边填 1

(i) 由于垂直, 填入 grid 的 item 都不相同, 而且每一行, 每一列填入的 "Entries", 左右都维持不同的元素。

(ii) 这个 Grid 的特性是, 同一列的 Item 来自 左边 拉丁方阵相同的元素, 例如第三列, 左边填 "2"。来自同一行的 右边 填 "1"。
(第 2 行)

(iii) 证明的概念是如下:



↓ 6"

If two items are collinear in grid (1) and grid (2), then we have (1) row and row, $\Rightarrow L_1 \neq L_3$; and (2) row and column $\Rightarrow L_2 \neq L_4$. $\rightarrow \leftarrow$



(*) Again, we would like to mention one more time that the grid design does provide a construction of d -disjunct matrices. For larger d , the construction is getting closer to the information lower bound.

(**) Grid designs can be applied to Group Testings with bounded pool size which is very important in practical applications.

Theorem (Well-known?)

If k is a prime power, there exist $k-1$ mutually orthogonal Latin squares of order k . (Use finite field $GF(k)$!)

For example, if $k=3$, there exist two orthogonal Latin squares of order 3. And if $k=9$, there are "8" MOLS(9).

$$(*) \quad n = k^2, \quad k=9, \quad 2(q-1) = 8, \quad q=5.$$

$\lceil \frac{d+1}{2} \rceil$ grids are enough for identifying d positives.

\Rightarrow " $d \leq 9$ " positives in 81 items can be identified by using at most $5 \cdot 2k = 90$ tests. ($2k$ tests for each array!)

⊙⊙ Arrange each line (test) as a row, we obtain a d -disjunct matrix. (Bounded row size!!)

Remark In general, $k^2 \ll n$ or $r \times c \ll n$.
(r rows and c columns)

In this situation, we have an $r \times c$ grid design. Again, it's a design problem or equivalently a graph decomposition problem. (Or a graph packing problem!)