GT 12 (12,13)

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(*) $|x-y| \le 1$ and $x+y \le 2^k$, let $x \le y$.

$\Rightarrow$ $y-x \le 1$ and $x+y \le 2^k$

$\Rightarrow$ $x \ge y-1$ thus $2y-1 \le 2^k$, $2y \le 2^k+1$.

If $y > 2^{k-1}$, then $2y \ge 2^k+2$. Hence $y \le 2^{k-1}$, so is x.

Lemma Let (a,b) be a state such that $b \ge a-1 \ge 1$. Then there exists a test in this state yielding states (a_1,b_1) and (a_2,b_2) such that

(1) $\lfloor \frac{a}{2} \rfloor \le a_1 \le \lfloor \frac{a+1}{2} \rfloor$, $\lfloor \frac{a}{2} \rfloor \le a_2 \le \lfloor \frac{a+1}{2} \rfloor$;

(2) $b_1 \ge a_1 - 1$, $b_2 \ge a_2 - 1$; and

(3) $ch(a_1,b_1)$, $ch(a_2,b_2) \le ch(a,b) - 1$.

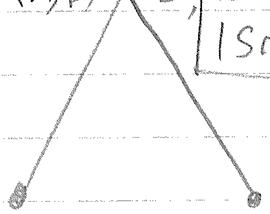
Proof.

Case 1. $b = a-1 \ge 1$

a is even, $a = 2c$

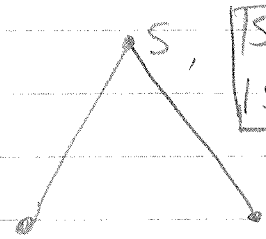
choose

(A,B) S, $\boxed{|S \cap A| = c \quad |S \cap B| = c}$



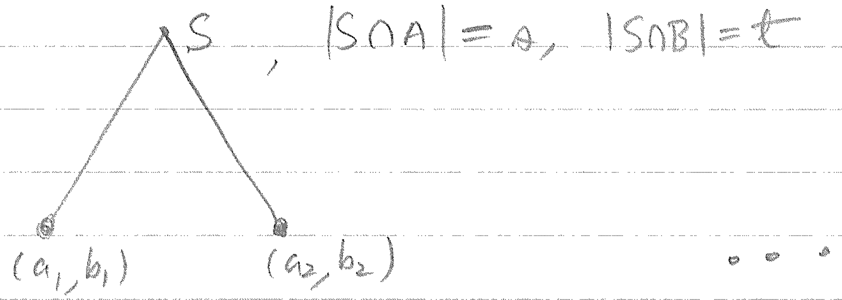
a is odd, $a = 2c+1$

S, $\boxed{|S \cap A| = c \quad |S \cap B| = c - \lfloor \frac{l}{2} \rfloor}$



$l = ch(a,a-1) \le c$

<u>Case 2</u> $b > a-1 \geq 1$



$|S \cap A| = a$, $|S \cap B| = t$

The idea of handling more lies, say "three".

Then, each state can be represented by a 4-tuple (a, b, c, d)

where $|A| = a$, $|B| = b$, $|C| = c$ and $|D| = d$.

A: $e \in A$ iff none of the answers is a lie (Truth set)

B: $e \in B$ iff exactly one of the answers is a lie (One lie set)

C: $e \in C$ iff exactly two of the answers are lies (Two lies set)

D: $e \in C$ iff exactly three of the answers are lies (Three lies set)

positive item (real)

Queries: "Is $e \in Q$?" where $Q = X \cup Y \cup Z \cup V \subseteq N$,

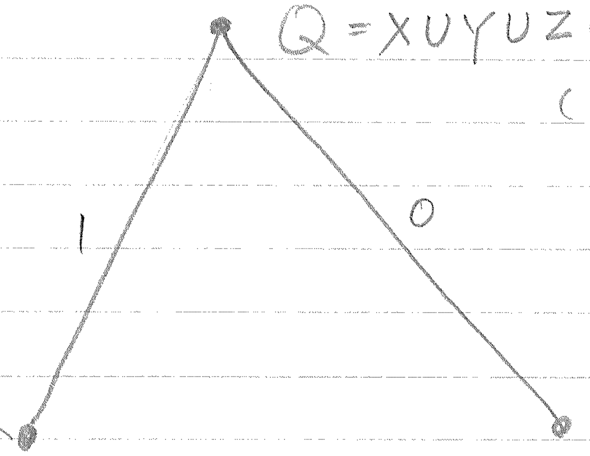
and $X \subseteq A$, $Y \subseteq B$, $Z \subseteq C$ and $V \subseteq D$.

(A, B, C, D) ← state of sets

$Q = X \cup Y \cup Z \cup V$

(Is $e \in Q$?)

↑ positive item



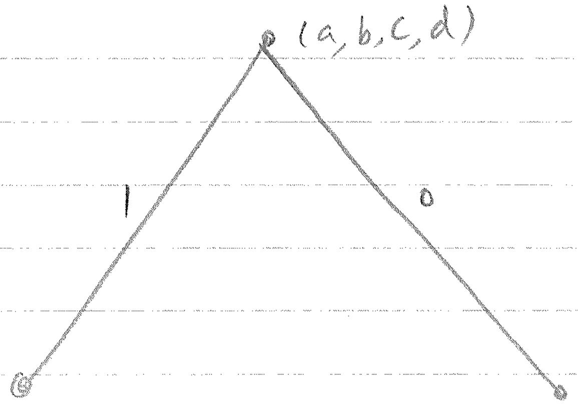
$(A \cap Q, (A \cap \bar{Q}) \cup (B \cap Q), (B \cap \bar{Q}) \cup (C \cap Q), (C \cap \bar{Q}) \cup (D \cap Q))$

$(A \cap \bar{Q}, (A \cap Q) \cup (B \cap \bar{Q}), (B \cap Q) \cup (C \cap \bar{Q}), (C \cap Q) \cup (D \cap \bar{Q}))$

(a, b, c, d)

$|X| = x, |Y| = y, |Z| = z, |V| = v$

$X \subseteq A, Y \subseteq B, Z \subseteq C, V \subseteq D.$



$(x, \underline{a-x}+y, b-\underline{y}+z, c-\underline{z}+v)$ $(a-x, \underline{x}+b-y, \underline{y}+c-z, \underline{z}+d-v)$

Weight of (a,b,c,d) : with $\boxed{k}$ more tests to do.

$$\boxed{\left(a \cdot \binom{k}{3} + b \cdot \binom{k}{2} + c \binom{k}{1} + d\right) = w_k(a,b,c,d).}$$

In truth set, there are three lies to come!

In lie sets, there are 2, 1, 0 lies to come!

Definition

The character of the state (a,b,c,d),

$$ch(a,b,c,d) = \min\{k \mid w_k(a,b,c,d) \le 2^k\}$$

Similar but more complicate arguments are needed in obtaining the answer.

So, find $M^3(1,10^6)$.

Please refer to :

A. Negro and Sereno,

1. Solution of Ulam's problem on binary search with three lies, JCT(A) 59 (1992), 149-154.
2. An Ulam's searching game with three lies, Advances in Applied Math., Vol. 13, 4, 1992, 404-428.

Two lies

W. Guzicki, Ulam's searching game with two lies, JCT(A) 54, 1990.

Now, we come back to consider non-adaptive algorithms.

Lies are corresponding with errors occurred in outcome vector.

If we have an error, then a $(1;3)$-disjunct matrix can be applied to find a positive with one error. So, if the Ulam's game is concerned, it is expected to find a $t \times 10^6$ matrix such that for any two columns C_i and C_j, $|C_i \setminus C_j| \geq 3$.

In case that we can find a positive in 25 tests, then it is expected that we can let $X = \mathbb{Z}_{25}$ and there are 10^6 subsets of $\mathbb{Z}_{25}$, such that for any two of them C_i and C_j, $|C_i \setminus C_j| \geq 3$.

<u>Exercise 1.</u> Can the above question be done? If not how many elements in X we need?

<u>Exercise 2.</u> In order to find a positive with 3 errors, then we need a $(1;7)$-disjunct matrix. Can we estimate how many tests we have in order to solve the problem $M^3(1,10^6)$?

The construction of (1;3)-disjunct matrices

Observation

1. Let $(X, \mathbb{B})$ be a block design: 2-$(v,4,1)$ design. Then we have a (1;3)-disjunct matrix: $v \times b$ (0,1)-matrix where $b = v(v-1)/12$.

2. If $v = 25$, then a 2-$(25,4,1)$ design exists. Hence, we have a 25×50 (1;3)-disjunct matrix.

3. If the block size we use in $\mathbb{B}$ is k, then two blocks in $\mathbb{B}$ can have at most $k-3$ elements in common.

So, if $X = \mathbb{Z}_{25}$, how many blocks we can get which satisfies property 3?