Lecture 14 June, 1.

Hadamard 3-designs

Before we consider Hadamard 3-designs, we define a notion called extension.

Definition (Extension)

If $(X, \mathbb{B})$ is a t-design where $t \geq 2$ and $(X', \mathbb{B}')$ is a $(t-1)$-design such that there exists an $x_0 \in X$ and $(X', \mathbb{B}')$ is isomorphic to $(X \setminus \{x_0\}, \mathbb{B}_{x_0})$ where $\mathbb{B}_{x_0} = \{B \mid B \cup \{x_0\} \in \mathbb{B}\}$, then $(X, \mathbb{B})$ is an extension of $(X', \mathbb{B}')$.

Definition (Derived)

If $(X, \mathbb{B})$ is an extension of $(X', \mathbb{B}')$, then $(X', \mathbb{B}')$ is called a derived design of $(X, \mathbb{B})$.

Open problem

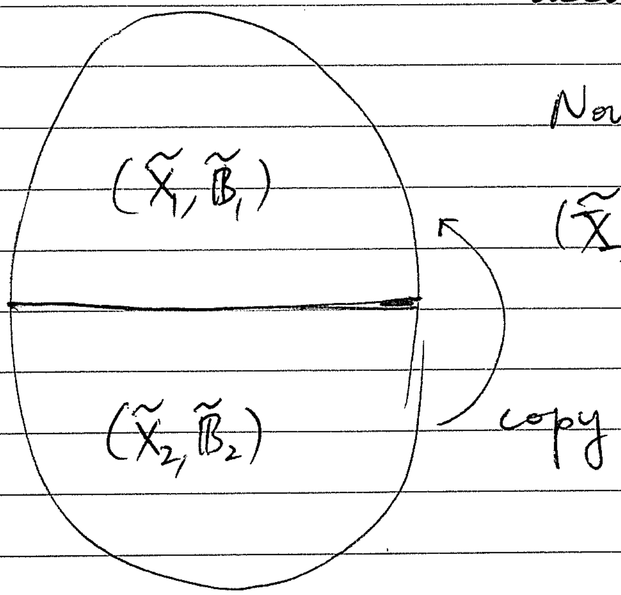
Is every Steiner triple system of order v a derived design of an Steiner quadruple system of order $v+1$?

- Every $\underset{\downarrow \atop \text{3-design}}{SQS(v+1)}$ is an extension of some $\underset{\downarrow \atop \text{2-design}}{STS(v)}$.

<u>Theorem 1</u> If $(X_1, \mathbb{B}_1)$ is a derived $STS(v)$ and $(X_1, \mathbb{B}_1)$ is a subsystem of $(X, \mathbb{B})$ (of order $2v+1$), then $(X, \mathbb{B})$ is a derived $STS(2v+1)$.

<u>Proof.</u> It suffices to prove that there exists an $SQS(2v+2)$, $(\tilde{X}, \tilde{\mathbb{B}})$ such that $(X, \mathbb{B})$ can be obtained by deleting one element from $\tilde{X}$. Moreover, let $X = X_1 \cup \tilde{X}_2$ and the 1-factorization used is $\mathcal{F}$.

Let $(\tilde{X}_1, \tilde{\mathbb{B}}_1)$ be the extension of $(X_1, \mathbb{B}_1)$ where $\tilde{X}_1 = X_1 \cup \{\infty\}$. Therefore, $(\tilde{X}_1, \tilde{\mathbb{B}}_1)$ is an $SQS(v+1)$. Now, we construct $(\tilde{X}, \tilde{\mathbb{B}})$ by using doubling construction with a subsystem $(\tilde{X}_1, \tilde{\mathbb{B}}_1)$ such that the 1-factorization used for $\tilde{X}_1$ is $\mathcal{G}$ and $\tilde{X}_2$ is $\mathcal{F}$.



Now, it's not difficult to check $(\tilde{X}, \tilde{\mathbb{B}})$ is an $SQS(2v+2)$ and

$$(\tilde{X} - \infty, \tilde{\mathbb{B}} \setminus \tilde{\mathbb{B}}_\infty) \cong (X, \mathbb{B}).$$

Here $\tilde{\mathbb{B}}_\infty = \{$ blocks containing ∞ in $\tilde{\mathbb{B}}\}$.

∎

Review that a Hadamard 2-design is a $2-(4\lambda+3, 2\lambda+1, \lambda)$ design. Now, we are ready to prove the following Theorem.

<u>Theorem 2</u> Let $(X, \mathbb{B})$ be a $2-(4\lambda+3, 2\lambda+1, \lambda)$ design. Then $(X, \mathbb{B})$ has an extension $(\tilde{X}, \tilde{\mathbb{B}})$ which is a $3-(4\lambda+4, 2\lambda+2, \lambda)$ design. <u>Ex. 3.3.</u> Prove Theorem 2 in more details. (10 points)

<u>Proof.</u> We prove a more general result: A $2-(4k+3, 2k+1, \lambda)$ design does have an extension to a $3-(4k+4, 2k+2, \lambda)$ design.

Clearly, by letting $k = \overset{2\lambda+1}{\checkmark}$, we have Theorem 2. $\quad b = \frac{\lambda(2k+1)2k}{k(k-1)}$

Now, assume that (X, B) is a $2-(2k+1, k, \lambda)$ design.

Let $\tilde{X} = X \cup \{\infty\}$. Let $\tilde{\mathbb{B}} = \{\underset{①}{B \cup \{\infty\} \mid B \in \mathbb{B}}\} \cup \{\underset{②}{X \setminus B \mid B \in \mathbb{B}}\}$.

Claim: $(\tilde{X}, \tilde{\mathbb{B}})$ is a $3-(2k+2, k+1; \lambda)$ design.

It suffices to show that any three distinct points (elements, varieties) are in precisely λ blocks of $\tilde{\mathbb{B}}$. First, if the three points contain ∞, then it follows from the fact $(X, \mathbb{B})$ is a $2-(4k+3, 2k+1, \lambda)$ design. On the other hand, let none of the three points is ∞. That, let x, y, z be three points in X. Then, there are two cases to consider, either $\{x,y,z\} \cap B = \phi$ or $\{x,y,z\} \cap B \neq \phi$ for each $B \in \mathbb{B}$. Let the number of blocks of ~~of the first type~~ be c_0 and ~~the # of blocks containg~~ x, y and z be c_1. Then, $c_0 + c_1 = b - 3r + 3\lambda$. (?)

For convenience, let c_{xy} be the number of blocks in $\mathbb{B}$ containing x and y but not z, c_x be the number of blocks in $\mathbb{B}$ containing x but not y or z, etc. Since the number of blocks containing x and y is $c_{xy} + c_{xyz} = \lambda$, $c_{yz} + c_{xyz} = c_{xz} + c_{xyz} = \lambda$. (2-design property).

Furthermore, $c_x + c_{xy} + c_{xyz} + c_{xz} = r$. This implies that

$$c_x = c_y = c_z = r - 2\lambda + c_{xyz}.$$ On the other hand,

$$b = c_0 + c_x + c_y + c_z + c_{xy} + c_{xz} + c_{yz} + c_{xyz}.$$ Hence

$$c_0 + c_{xyz} = b - c_x - c_y - c_z - c_{xy} - c_{xz} - c_{yz} = b - (c_x + c_{xz}) - (c_y + c_{yx}) - (c_z + c_{yz})$$

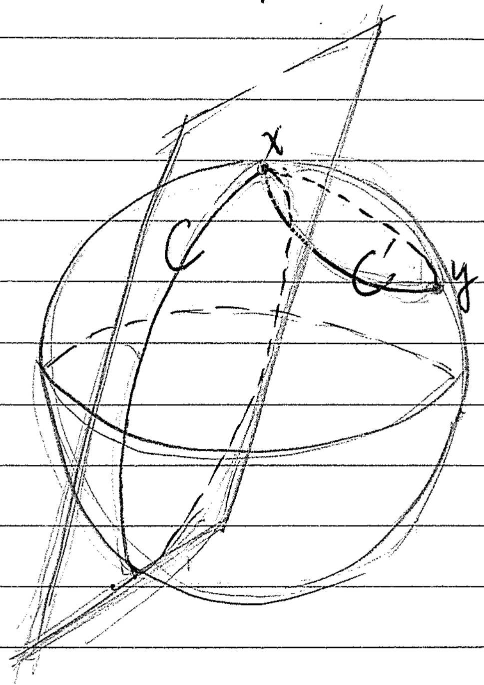
$$= b - 3(r - \lambda) = b - 3r + 3\lambda.$$

Note that in $\tilde{\mathbb{B}}$, x, y, and z occur in $\underset{①}{c_0} + \underset{②}{c_{xyz}}$ blocks.

Finally, $b - 3r + 3\lambda = \dfrac{\lambda(2k+1)2k}{k(k-1)} - 3\lambda \dfrac{2k}{k-1} + 3\lambda$

$$= \frac{\lambda}{k(k-1)}\left[2k(2k+1) - 6k^2 + 3k(k-1)\right] = \lambda.$$ ∎

Inversive plane



Can $2-(n^2,n,1)$ Affine plane of order n be extended to a $3-(n^2+1, n+1, 1)$ design?

N.C.
If $t-(v,k,\lambda)$ design has an extension, then $k+1 \mid b\cdot(v+1)$.

- In $2-(n^2,n,1)$ design, $b = \frac{n^2(n^2-1)}{n\cdot(n-1)} = n\cdot(n+1)$.
- $k+1 = n+1 \mid b\cdot(n^2+1)$. ✓

Definition (Inversive plane) (Also known as a Möbius plane)
$(X, \mathbb{B})$ is an inversive plane if ← the set of circles (blocks)

(1) Any three distinct points of X are in exactly one common circle;

(2) If x and y are two points of X and C is a circle $\in \mathbb{B}$ with $x \in C$ but $y \notin C$, then there is a unique circl $C' \in \mathbb{B}$ such that $\{x,y\} \subseteq C'$ and $C \cap C' = \{x\}$; and

(3) There exist four points which are not on a common circle.

Example

Take X as the set of $(x,y,z) \in \mathbb{R}^3$ such that $x^2+y^2+z^2=1$.

Let $\mathbb{B}$ be the set of circles obtained by the intersection of a plane and the surface of the sphere $\{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$ (with more than two points).

(1), (2) and (3) are clearly true.

Theorem 3 If $(X, \mathbb{B})$ is an inversive plane and $x_0 \in X$, then $\mathbb{B}_{x_0} = \{ B \setminus \{x_0\} \mid x_0 \in B \in \mathbb{B}\}$ together with $X \setminus \{x_0\}$ is an affine plane. ($(X, \mathbb{B})$ is an extension of an Affine plane (in finite case))

Proof. By definition of an Affine plane.

Theorem 4 The finite inversive planes are exactly the class of 3-designs with parameters $(n^2+1, n+1, 1)$ with $n \ge 2$.

(•) For every prime power q, there exists an inversive plane with $q+1$ points on a circle, namely $\mathcal{S}_2(q)$.

$n=3$, $SQS(10)$.

Ex. 3.4. Find an inversive plane 3-(17, 5, 1). (10 points)

You may look for the answer anywhere you can find.