Lecture 12 May 9, '11

NO. 1 DATE

Projective Geometry

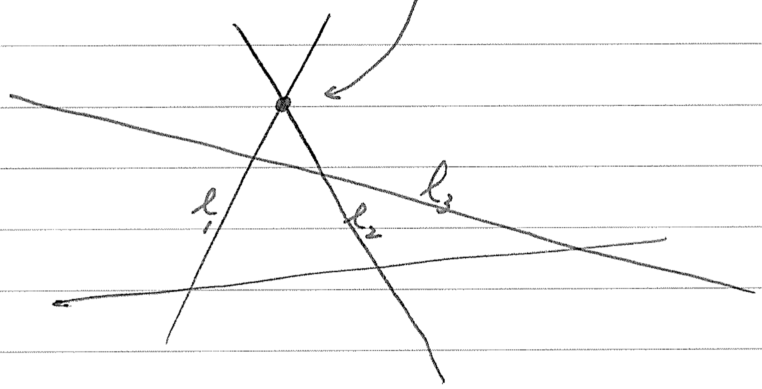
A projective geometry $\mathcal{P}$ is a (not necessarily finite) structure of points and lines such that
(varieties) (blocks)

(P1) every pair of points are on a unique common line;

(P2) every line contains at least three points;

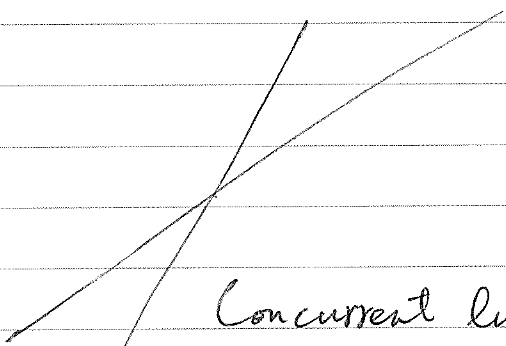
(P3) $\mathcal{P}$ contains a set of three points which are not on a common line;

(P4) if a line intersects two sides of a triangle but not contain their common point, then it intersects the 3rd side.

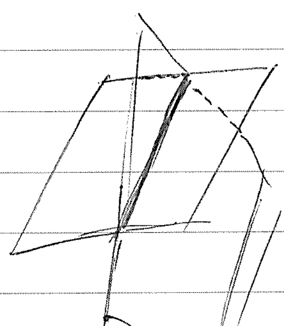


(P5) A projective plane is a projective geometry with this extra condition: every pair of distinct lines contain a common point.

- Let V be an $n+1$-dim. vector space over $GF(q)$ where q is a prime power. The "projective geometry" $PG(n,q)$ is the geometry whose points, lines, planes, ... are 1-, 2-, 3-, ... dimensional subspace of V. ($PG(2,q)$ is a projective plane.)
- In general, a $(k+1)$-dimensional subspace will be called a k-flat, so the point is a 0-flat, a line is a 1-flat, ..., etc. All the flats are "varieties" in general term.
- Two varieties are "incident" if one contains the other, for example, a point on a line, a line on a plane, ..., etc.
- Two varieties are "concurrent" if their intersection is non-empty. (Say, two lines contain a common point.)



Concurrent lines



Concurrent planes

Vector spaces over finite fields

- In the construction of good designs and correspondingly good codes, finite fields play an important role.
- Since we are interested in linear codes, not surprisingly, vector spaces are the center of study.

Definition (Linear codes)

A code $\mathcal{C}$ of length n (defined on F_q) is a linear code if $\mathcal{C}$ is a linear space of the n-dimension vector space F_q^n. (Review that a vector space has two operations: vector sum and scalar product.)

- A binary code is a design by taking the set of supports of all binary codewords.

(*) A linear code corresponds to a design with larger "λ".

(**) Of course, we may not have a balanced design or a pairwise balanced design when we use a linear code.

Proposition (Well-known property)

The number of k-dim. subspace of an n-dim. vector space over $GF(q)$ is the Gaussian coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} \frac{q^n - q^i}{q^k - q^i} .$$

(Note that q can also be positive real number except 1 when $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is used.) $\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$ (?)

Observation

- If we have a 2-dim. (n) vector space V over $GF(q)$, then V contains q^2 (q^n) vectors.

(1) • There are $(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})$ distinct bases in a vector space of dimension n. (?)

(2) • There are $(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})$ distinct bases in V which generate a k-dim. subspace.

Combining (1) and (2), we have proved the above proposition.

<u>Proposition</u> A

Let V be a vector space of dimension n over $GF(q)$ and let U be an m-dim subspace of V, $m \geq 0$. Then, the number of $(m+h)$-dimension subspaces containing U is

$$\frac{(q^{n-m}-1)(q^{n-m}-q)\cdots(q^{n-m}-q^{h-1})}{(q^h-1)(q^h-q)\cdots(q^h-q^{h-1})}.$$

<u>Proof.</u> By the idea of counting bases we have the number:

$$\frac{(q^n-q^m)(q^n-q^{m+1})\cdots(q^n-q^{m+h-1})}{(q^{m+h}-q^m)(q^{m+h}-q^{m+1})\cdots(q^{m+h}-q^{m+h-1})}.$$

This implies the above result. ∎

To consider the construction of projective geometry, in what follows, assume that V is a vector space of dimension $n+1$ over $F = GF(q)$.

(Instead of "n"-dim.)

$F = GF(q)$

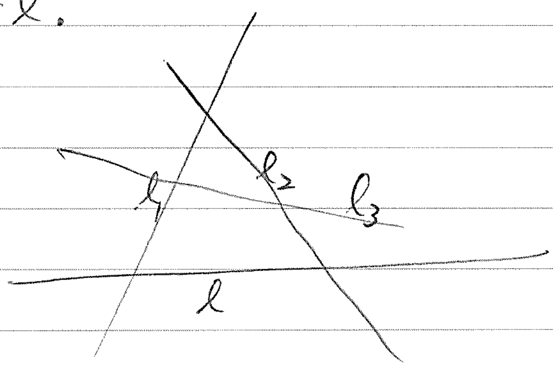
<u>Definition</u> ($P_{n,i}(F)$)

on V ($(n+1)$-dim. v.sp.)

We define $P_{n,i}(F)$ to be the structure whose points are the 0-flats and whose blocks are the i-flats.

- $P_{n,1}(F)$ is a <u>projective geometry</u> provided $n>1$.

<u>proof.</u> Clearly, two independent vectors will determine a unique 1-flat, (P1) holds. Since $n>1$, V is of at least dimension 3, hence each line (1-flat) contains at least 3 points. (P2). By the same reason, (P3) holds. If (P4) is not true, then l_3 will be the same line as l.



■

For convenience, we also use $P_{n,i}(q)$ for $P_{n,i}(F)$ if $F=GF(q)$.

<u>Proposition</u> $P_{n,1}(q)$ is a 2-$\left(\frac{q^{n+1}-1}{q-1}, \frac{q^2-1}{q-1}, 1\right)$ design.

If $n>i>1$, then we also have a 2-design except the λ is going to be very large.

<u>Proposition</u> $\mathcal{P}_{n,i}(q)$ is a $2-\left(\frac{q^{n+1}-1}{q-1}, \frac{q^{i+1}-1}{q-1}, \prod_{j=0}^{j=i-2} \frac{q^{n-1-j}-1}{q^{j+1}-1}\right)$.

<u>Proof.</u> Consider U a 2-dim. subspace of V (dim. $n+1$) and count the number of $(i+1)$-dim. subspace which contain U.

By Proposition A, the number is

$h+2=i+1$
$\Rightarrow h=i-1$

$$\frac{(q^{n+1-2}-1)(q^{n+1-2}-q)\cdots(q^{n+1-2}-q^{i-2})}{(q^{i-1}-1)(q^{i-1}-q)\cdots(q^{i-1}-q^{i-2})}$$

$$=\frac{(q^{n-1}-1)(q^{n-2}-1)\cdots(q^{n-i+1}-1)}{(q^{i-1}-1)(q^{i-2}-1)\cdots(q-1)}.$$ ■

<u>Affine Geometry</u>

<u>Definition</u> An affine geometry is obtained from a projective geometry by deleting a fixed hyperplane and all its subspace.

<u>Affine plane</u>

An affine plane A (not necessarily finite) is a structure of points and lines such that

(A1) every pair of distinct points are on a unique common line;

(A2) if (p, y) is a non-incident point-line pair, then there is a unique line z containing p which does not meet y; and

(A3) A contains a set of three points not on a common lin

<u>Definition</u> Let $F = GF(q)$.

$A_{n,i}(q)$ is the structure whose points are points in general and whose blocks are i-flats. If $i = n-1$, then the i-flats are hyperplanes.
↓ q-dimension

<u>Proposition</u> $A_{n,i}(q)$ is a $2\text{-}\left(q^n, q^i, \prod_{j=0}^{j=i-2} \frac{q^{n-1-j}-1}{q^{j+1}-1}\right)$ design if $i \geq 2$ and $A_{n,1}(q)$ is $2\text{-}(q^n, q, 1)$ design, this is known as the projective plane of order q.

How about 3-design?

Fact: $P_{n,i}(q)$ is never a 3-design. (三点共线与不共线出现的次数不一樣!)

$A_{n,i}(q)$: each line contains q points.

- By the same reason as $P_{n,i}(q)$, $A_{n,i}(q)$ will not be a 3-design if $q \geq 3$. So, the only possibility is $q=2$. In fact, we have

<u>Proposition</u> $A_{n,i}(2)$ is a 3-$\left(2^n, 2^i, \prod_{j=0}^{i-3} \frac{2^{n-2-j}-1}{2^{j+1}-1}\right)$ design provided $n > i \geq 3$ and that $A_{n,2}(2)$ is a 3-$(2^n, 4, 1)$ design.