# Lecture 5 Special Squares – Continued

One of the most useful Latin squares in applications is called a Latin square with hole.

**Definition**

Let $\mathbb{Z}_n = \dot{\bigcup}_{i=1}^{k} H_i$. We say: $L = [l_{ij}]$ is Latin square with hole of type $\prod_{i=1}^{k} h_i$ where $h_i = |H_i|$, provided the following two conditions hold:

(1) $\forall\, l \in \{1, 2, \cdots, k\}$ and $i, j \in H_l$, $l_{ij}$ is empty; and

(2) $\forall\, x \in \mathbb{Z}_n$, $x$ occurs in each row and each column at most once, furthermore, if $t \in H_l$, then $t$ can not occur in the $t$-th row and $t$-column.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | | | 8 | 5 | 4 | 7 | 6 | 3 |
| 2 | | | 6 | 7 | 8 | 3 | 4 | 5 |
| 3 | 8 | 6 | | | 7 | 2 | 5 | 1 |
| 4 | 5 | 7 | | | 1 | 8 | 2 | 6 |
| 5 | 4 | 8 | 7 | 1 | | | 3 | 2 |
| 6 | 7 | 3 | 2 | 8 | | | 1 | 4 |
| 7 | 6 | 4 | 5 | 2 | 3 | 1 | | |
| 8 | 3 | 5 | 1 | 6 | 2 | 4 | | |

For convenience, we use $\{1, 2, \cdots, n\} = \mathbb{Z}_n$

(*) A <u>commutative</u> LS(8) with hole of type $2^4$.

**Proposition 1.** For odd $n$, a commutative $LS(2n)$ with hole of type $2^n$ exists.

*Proof.* Let $M$ be an idempotent commutative $LS(n)$. Then let $L = \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} \otimes M$. By deleting all $2\times 2$ subsquares along the diagonal, we have the desired square. $\blacksquare$

$$\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 0 & 2 & 1 \\ \hline 2 & 1 & 0 \\ \hline 1 & 0 & 2 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|}\hline 00 & 01 & 20 & 21 & 10 & 11 \\ \hline 01 & 00 & 21 & 20 & 11 & 10 \\ \hline 20 & 21 & 10 & 11 & 00 & 01 \\ \hline 21 & 20 & 11 & 10 & 01 & 00 \\ \hline 10 & 11 & 00 & 01 & 20 & 21 \\ \hline 11 & 10 & 01 & 00 & 21 & 20 \\ \hline \end{array}$$

$00 \to 0$
$01 \to 1$
$10 \to 2$
$11 \to 3$
$20 \to 4$
$21 \to 5$

**Theorem 2.** For all $n \geq 3$, a commutative $LS(2n)$ with hole of type $2^n$ exists.

*Proof.* It suffices to prove the case when $n$ is even. (Proposition 1 is true.) Since the details are very tedious, we sketch the proof. First, we construct small orders for $n = 4, 6$. Then, we embed a commutative $LS(2m)$ with hole of Type $2^m$ in a commutative $LS(4m+4)$ with hole of type $2^{2m+2}$. Therefore, for $2n = 4l$ (when $n$ is even), the square is obtained by embedding of a square of order $2l-2$ into a square $2(2l-2)+4 = 4l$. ■

Reference: Chin-Mei Fu and Hung-Lin Fu, On the intersection of Latin squares with holes, Utilitas Mathematica, Vol. 35, 1989, 67–74.

Bonus (10 points) Construct a commutative $LS(12)$ with holes of type $2^6$.

Proposition 3 A Latin square of order $km$ with hole of type $k^m$ exists for each $m \geq 3$.

Proof. Since an idempotent $LS(m)$, $M$, exists for each $m \geq 3$, the proof follows by the existence of $K \otimes M$ where $K$ is a $LS(k)$. ■

(Here, commutative law is not required.)

(*) If we do a similar square which is commutative, then the proof is harder, but it does exist $k$ and $m$ are larger than 2.

Definition (Complete $LS(n)$)

An $LS(n)$ is called horizontally complete (row-) (resp. vertically (column-)

complete) if for any ordered pair $(\alpha,\beta)$ in $\mathbb{Z}_n^2 \setminus \{(i,i) \mid i \in \mathbb{Z}_n\}$, there exists a row (resp. column) of the square in which $\alpha$ and $\beta$ occur next to each other following the order of $\alpha$ and $\beta$.

| | | | |
|---|---|---|---|
| 0 | 1 | 2 | 3 |
| 1 | 3 | 0 | 2 |
| 2 | 0 | 3 | 1 |
| 3 | 2 | 1 | 0 |

A <u>complete</u> Latin square of order 4.

Furthermore, if $L$ is both horizontally and vertically complete, then $L$ is a complete Latin square.

In general, constructing a complete Latin square is not easy at all. But, if we only consider row-complete or column-complete (not necessarily be both), then it is easier.

<u>Proposition 4.</u> If $n$ is even, then there exists a row-complete (column) Latin square of order $n$.

<u>Proof.</u> Let $n=2m$. The proof follows by constructing the first row (or column) and then adding $k \pmod{n}$ to the $k+1^{th}$ row (or column

Now, the first row (or column) is given by

$$(0, 2m-1, 1, 2m-2, 2, 2m-3, \cdots, m-1, m).$$

So, we have (or its transpose) the desired square since all the differences are distinct (from 1 to $2m-1$). ∎

Example, $m=4$

complete

An complete $LS(8)$ can be obtained by permuting its rows,

$$\left\{\begin{array}{cccccccc}
0 & 7 & 1 & 6 & 2 & 5 & 3 & 4 \\
1 & 0 & 2 & 7 & 3 & 6 & 4 & 5 \\
2 & 1 & 3 & 0 & 4 & 7 & 5 & 6 \\
3 & 2 & 4 & 1 & 5 & 0 & 6 & 7 \\
4 & 3 & 5 & 2 & 6 & 1 & 7 & 0 \\
5 & 4 & 6 & 3 & 7 & 2 & 0 & 1 \\
6 & 5 & 7 & 4 & 0 & 3 & 1 & 2 \\
7 & 6 & 0 & 5 & 1 & 4 & 2 & 3
\end{array}\right.$$

← row complete (column complete can be obtained by finding its transpose)

The following idea was obtained by B. Gordon in 1961.

Theorem 5. If there exists a finite group of order $n$, $G = \{a_1, a_2, \cdots, a_n\}$ such that $a_1, a_1a_2, a_1a_2a_3, \cdots, a_1a_2\cdots a_n$ are $n$ distinct elements in $G$, then there exists a complete Latin square of order $n$.

Proof. Let $b_i = a_1a_2\cdots a_i$, $i = 1, 2, \cdots, n$. Now, construct an $LS(n)$, $L = [l_{ij}]$ where $l_{ij} = b_i^{-1}b_j$. Now, we can check the following properties of $L$.

(1) Since $G$ is a group, $L$ is indeed a Latin square of order $n$. (?)

(2) $L$ is row-complete.

It suffices to show that for any two distinct elements $\alpha$ and $\beta$ in $G$, there exist $s$ and $t$ such that $\alpha = b_s^{-1} b_t$ and $\beta = b_s^{-1} b_{t+1}$. We claim that $s$ and $t$ can be found uniquely. Since $\alpha = b_s^{-1} b_t$ and $\beta = b_s^{-1} b_{t+1}$, $\beta = \alpha \cdot a_{t+1}$. So, clearly, $a_{t+1} = \alpha^{-1} \cdot \beta$ which gives ~~as~~ unique $t$ for the solution. Now, by $\alpha = b_s^{-1} b_t$, $s$ is uniquely determine since $t$ is fixed.

(3) $L$ is column-complete.

Similarly, we try to find $s$ and $t$ such that $b_s^{-1} b_t = \alpha$ and $b_{s+1}^{-1} b_t = \beta$. Hence $\alpha^{-1} = b_t^{-1} b_s$ and $\beta^{-1} = b_t^{-1} b_{s+1}$, thus $\beta^{-1} = \alpha^{-1} \cdot a_{s+1}$. Now, the proof follows as the argument above. ∎

(*) <u>Exercise 1.9.</u> Construct a complete Latin square of even
(Bonus)
order by using Theorem 5.

Definition (Diagonal L.S.).

A Latin square $L = [l_{i,j}]$ is called a diagonal Latin square, if $\{l_{i,i} \mid 1 \le i \le n\} = \mathbb{Z}_n = \{l_{i,n-i+1} \mid 1 \le i \le n\}$.

| | | | |
|---|---|---|---|
| 0 | 1 | 2 | 3 |
| 2 | 3 | 0 | 1 |
| 3 | 2 | 1 | 0 |
| 1 | 0 | 3 | 2 |

A diagonal Latin square of order 4

Proposition 6. For each $n \ge 4$, there exists a diagonal Latin square of order $n$.

Problem. Can you construct such squares?

Theorem 7. (Du et. al) For each $n \ge 4$ and $n \ne 6$, there exists a pair of orthogonal Latin squares of order $n$.

Corollary 8. For each $n \ge 3$, there exists a magic square of order $n$.

| | | |
|---|---|---|
| 4 | 3 | 8 |
| 9 | 5 | 1 |
| 2 | 7 | 6 |

Proof. $n=3$ can be obtained by direct construction.

For $n \geq 4$, let $L$ and $M$ be two orthogonal diagonal Latin squares of order $n$. Now, let $Q$ be obtained by superimposing $L$ and $M$, i.e., $Q = [q_{i,j}]$ where $q_{i,j} = (l_{i,j}, m_{i,j})$. If we replace $q_{i,j}$ by $n \cdot l_{i,j} + m_{i,j} + 1$, then $Q$ is a magic square.

For example,

$$\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline 2 & 3 & 0 & 1 \\ \hline 3 & 2 & 1 & 0 \\ \hline 1 & 0 & 3 & 2 \\ \hline \end{array} \quad \perp \quad \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline 3 & 2 & 1 & 0 \\ \hline 1 & 0 & 3 & 2 \\ \hline 2 & 3 & 0 & 1 \\ \hline \end{array}$$

The sum is equal to
$$n \cdot (0+1+\cdots+n-1) + (0+1+\cdots+n-1) + n$$
$$= n \cdot n \cdot \frac{n-1}{2} + \frac{n-1}{2} \cdot n + n$$
$$= \frac{n(n-1)(n+1)}{2} + n$$
$$= \frac{n(n^2+1)}{2}.$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 6 & 11 & 16 \\ \hline 12 & 15 & 2 & 5 \\ \hline 14 & 9 & 8 & 3 \\ \hline 7 & 4 & 13 & 10 \\ \hline \end{array}$$

$n = 4$
(Sum $= 34$)

This is known as a magic square of order 4.

(*) This solution also answers a long-stand problem of finding a systematic construction for magic squares.