Definition 5.

1. Pebbling move

   Suppose p pebbles are distributed onto the vertices of a graph G. A **pebbling move**(step) consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex.

2. Factor

   Given a graph (nultigraph, general graph) G, we say that H is a **factor** of G if H is a spanning subgraph of G. A factor that is k-regular is called a **k-factor**.

3. k-factor

   Let G be a multigraph possibly with loops and k, a non-negative, integer-valued function on $V(G)$. Then a spanning subgraph H of G is called an **k-factor** of G if $\deg_H(v) = k$, for all $v \in V(G)$.

4. k-factorization and k-factorable

   - A **k-factorization** partitions the edges of the graph into disjoint k-factors.
   - A graph G is said to be **k-factorable** if it admits a k-factorization.

5. Length, Closed, and Circuit

   - The **length of a walk** is the number of edges (counting repetitions).
   - A walk is **closed** if the initial vertex is also the final vertex; otherwise, it is **open**.
   - A **circuit** is a closed trail with no repeated edges.

6. Clique number

   The **clique number** of a graph G, denoted $\omega(G)$, is the maximum number of vertices in a complete subgraph of G.

7. Clique cover

   A **clique cover** of a graph G is a collection of cliques that contains every vertex of G.

8. Partite sets

   - A simple graph or multigraph is **bipartite** if its vertices can be partitioned into two sets(called **partite sets**) in such a way, that no edge joins two vertices in the same set.
   - A **complete bipartite graph** is a simple bipartite graph in which each vertex in one partite set is adjacent to all the vertices in the other partite set.

9. Complete k-partite graph, Complete multipartite graph, and Balance complete multipartite graph

   A **complete k-partite graph** is a simple k-partite graph in which two vertices are adjacent if and only if they are in different partite sets. All such graph are called **complete multipartite graphs**. A **balance complete multipartite graph** is the complete k-partite graph which all the order of partite sets are equal.

10. Turán graph

    Let $n \geq r \geq 2$ be integers. The **Turán graph** $T_r(n)$ is the complete r-partite graph whose partite sets are of nearly equal sizes. By the other way, if $n = rs + t$ $(0 \leq t \leq r - 1)$, then $T_r(n)$ has t partite sets of cardinality $\lceil n/r \rceil$ and $r - t$ partite sets of cardinality $\lfloor n/r \rfloor$.

11. Root, Ancestor, Leave and Plane tree

    - In a tree, let a vertex r be a **root**, then the tree is called a **rooted tree**. If $p(v)$ is the unique v, r-path in the tree, then the **parent** of v is the neighborhood of v in $p(v)$. And the other neighborhood of v are its **children**. The **ancestor** of v is all the vertices without v in the path $p(v)$. And, if $p(u)$ includes v, then u is a **descendent** of v. Vertices without descendent is called a **leave**.
    - A **rooted plane tree** or **plan tree** is a rooted tree which the children of every vertex have order from left to right.

12. Binary tree, Child, Subtree and k-ary tree

    - A **binary tree** is a rooted plan tree in which each vertex has at most two children, and each child is designated either a **left child** or a **right child**.

- The **left (right) subtree** of a vertex v in a binary tree T is the binary subtree spanning the left (right)-child of v and all of its descendants.
- A **k-ary tree** is a rooted plane tree in which each vertex has at most k children.

13. Circumference

    The **circumference** of a graph is the length of any longest cycle in a graph.

14. Odd and Even vertices

    - A vertex in a graph is said to be an **odd(even) vertex** if its vertex degree is odd(even).
    - An undirected graph is odd(even) if every vertex has odd(even) degree.

15. Deficiency

    The **deficiency** of graph is defined by $def(G) = \Sigma_{v \in V(G)}(\Delta(G) - \delta(v))$.

16. Loop, Multiple arc, and Simple digraph

    For a directed graph, a **loop** adds one to the in degree and one to the out degree. A **multi-arc** is a set of two or more arcs having the same tail and same head. A **simple digraph** is a digraph with no self-loops and no multi-arcs.

17. Directed path and Directed cycle

    - If a **directed path** leads from x to y, then y is said to be a successor of x and reachable from x, and x is said to be a predecessor reachable to y.
    - A **directed cycle** graph is a directed version of a cycle graph, with all the edges being oriented in the same direction.

18. Underlying graph

    The **underlying graph** of diagraph is the graph obtained by replacing each arc of diagraph by corresponding (undirected) edge.

19. Out-tree and In-tree

    A digraph **out-tree** is a tree having a root of indegree 0 and all other vertices of indegree 1, and an **in-tree** is an out-tree with all edges reversed.

20. Subgraph, Isomorphism, Decomposition, Union, Intersection, Adjacency matrix, and Incidence matrix

    - Definition of **subgraph**, **isomorphism**, **decomposition**, **union**, and **interesction** in digraph is the same to graph.
    - **Adjacency matrix** $A(D) = (a_{ij})$, where D is a digraph, a_{ij} is the number of arcs whose v_i is tail, and v_j is head in D.
    - **Incidence matrix** $M(D) = (m_{ij})$, where D is a simple digraph, $m_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is the tail of } e_j, \text{ and} \\ -1, & \text{if } v_i \text{ is the head of } e_j. \end{cases}$

21. Directed walk, Directed trail and Directed circuit

    A **directed walk** in a digraph G is an alternating sequence

$$W = v_0, e_1, v_1, e_1, \ldots, e_n, v_n,$$

    of vertices and arcs, such that $\text{tail}(e_i) = v_{i-1}$ and $\text{head}(e_i) = v_i$, for $i = 1, \ldots, n$. If every arc does not repeat, then the directed walk is also called **directed trail**. A closed directed trail is a **directed circuit**.

22. Eulerian digraph

    A digraph $D = (V, E)$ is **Eulerian** if and only if D is connected and for each of its vertices v, $d^-(v) = d^+(v)$. **Euler trail** is the directed trail contains all the arcs. A closed Euler trail is an **Euler circuit**.

23. Strict

    A digraph is **strict** if it has no loops, and no two directed edges have the same endpoints.

24. Extremal graph and Forbidden graph

    Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of families of graphs, and let $\Phi(n, F_n)$ be the set of graphs $G(n)$ that are H-free for every $H \in F_n$.

- The function $ex(n, F_n)$ of n is called the **extremal function** of the sequence $\{F_n\}$.
- The graphs $G \in \Phi(n, F_n)$ for which $e(G) = ex(n, F_n)$ are called **extremal graphs**.
- In this context, the families $\{F_n\}_{n=1}^{\infty}$ are called **forbidden graphs**.