Domination problem: Dec 8,

Definition (Dominating set)

Let G be a graph. A subset of $V(G)$, S, is a dominating set of G if for each vertex $v \in V(G) \setminus S$, v is adjacent to a vertex of S.

Definition (Domination number)

The domination number of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G, i.e.,
$\gamma(G) = \min.\{|S| \mid S \text{ is a dominating set of } G\}$.

Fact 1 $\gamma(K_n) = 1$, $\gamma(C_n) = \lceil \frac{n}{3} \rceil$ and $\gamma(K_{m,n}) = 2$ provided $m \geq n \geq 2$.

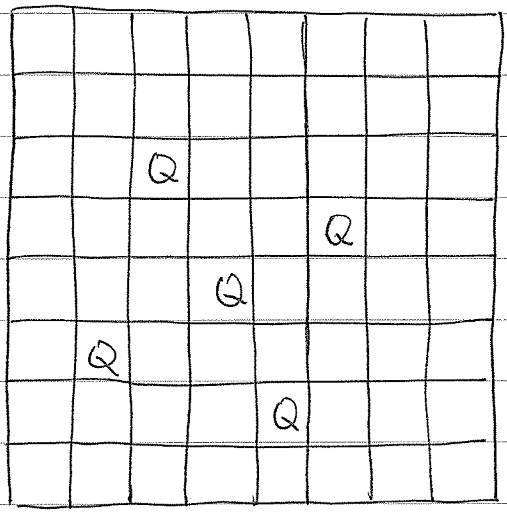
Fact 2 If there exist t vertices $v_1, v_2, \cdots, v_t$ such that for any $1 \leq i \neq j \leq t$, $N[v_i] \cap N[v_j] = \phi$, then $\gamma(G) \geq t$.

(Note: In G, the maximum "t" defined above is called the packing number of G, denoted by $\rho(G)$.)

__Fact 3__ $\gamma(G) \geq \rho(G)$.

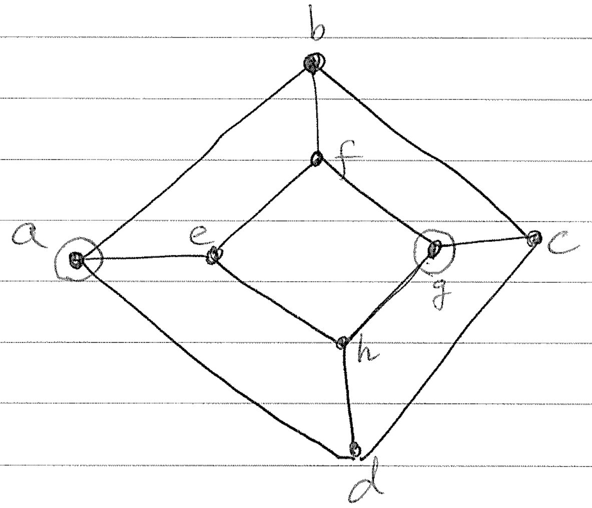
__Proposition__ If $\delta(G) \geq 2$, then $\gamma(G) \leq \lfloor \frac{n}{2} \rfloor$.
(G contains no isolated vertices.)

__Proof.__ Let D be a dominating set of G such that $|D| = \gamma(G)$. Now, consider $W = V(G) \setminus D$. For each vertex $v \in D$, v is adjacent to a vertex in W since D is a minimal dominating set. Hence, W is a dominating set of G and therefore $|W| \geq |D|$. This concludes the proof. ∎



Five-queens problem

(*) Five queens can dominate the chessboard.

G: 

$\{a, g\}$ is a dominating set of G.

Dominating sets of G can be obtained by using generating functions:

$(a+b+e+d)(b+a+f+c)(c+b+g+d)(d+a+h+c)(e+a+f+h)\cdot$

$(f+b+e+g)\cdot(g+c+f+h)\cdot(h+d+e+g)$

$= a^4\cdot g^4 + b^4 h^4 + b^3\cdot f\cdot h^4 + \cdots$ (Many Terms)

...

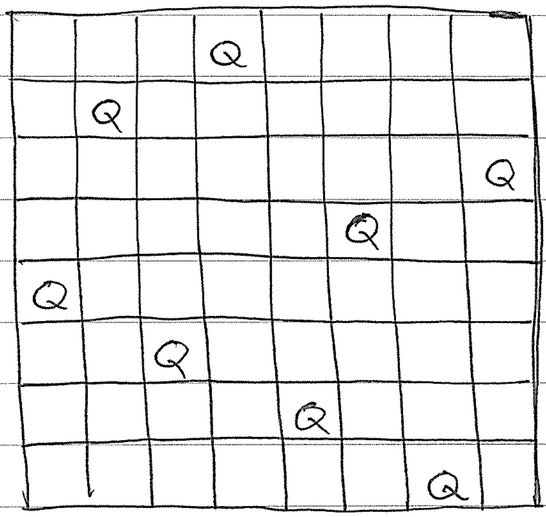
Each of them provides a dominating set ! (?)

Relationship between $\alpha(G)$ and $\gamma(G)$

Proposition

If S is a maximal independent set of G, then S is a dominating set of G. Moreover, $\alpha(G) \geq \gamma(G)$.

Proof. Since S is a maximal independent set, for each $v \in V(G) \setminus S$, v is adjacent to a vertex in S. This implies that S is a dominating set and thus $|S| \geq \gamma(G)$. By choosing S such that $|S| = \alpha(G)$, the proof follows. ∎



Independent 8 queens on a chessboard.

Bonus Problem (2 points, Due 12, 14, 15:00)

Let G be a graph defined on $\mathbb{Z}_{101}$. Two vertices i and j are adjacent if and only if $i-j \in \{1, 5, 6, -1, -5, -6\} \pmod{101}$. Find $\gamma(G)$.

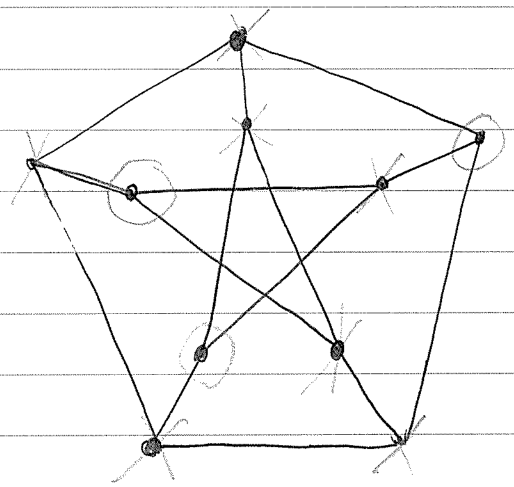
Remark There are quite a few results in the study of domination problem. You may try the following problem for challenge!

Proposition

Let G be a graph with p vertices. Then $\gamma(G) \geq 2\sqrt{p} - \Delta(G) - 1$.

(*) This is a good lower bound for many graphs.

$\chi(G) = 4$,



$\alpha(G) = 4$

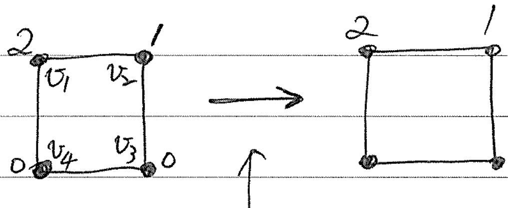
$\gamma(G) = ?$
$\parallel$
3

Pebbing Problem

(*) This is a "generalized" version of domination problem.

Definition (Distribution configuration)

A distribution configuration of pebbles (D.C.P.) on a graph G, denoted by δ_G, is a mapping $\delta_G : V(G) \longrightarrow \mathbb{N} \cup \{0\}$. The total pebbles used in a D.C.P. of G is denoted by δ^G.

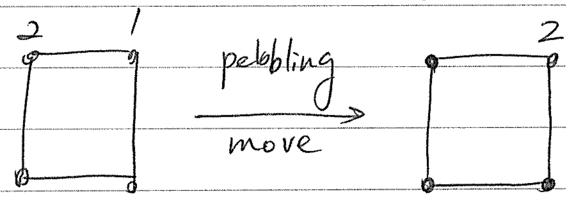


Delete "0".

Example: $\delta_G(v_1) = 2$, $\delta_G(v_2) = 1$, $\delta_G(v_3) = \delta_G(v_4) = 0$.

Definition (Pebbling move)

In a distribution configuration of pebbles on a graph G, a pebbling move consists of taking two pebbles from one vertex to an adjacent vertex with a missing pebble.

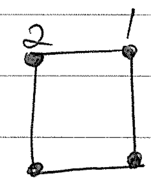


Definition (Pebbling)

A D.C.P. of G is a pebbling of G if for each vertex $v \in V(G)$, there exists a sequence of pebbling move such that one pebble can be moved to v.

Example

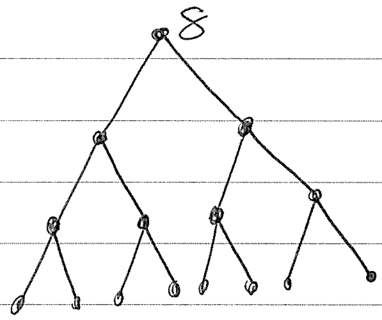
G_1 :



is a pebbling of C_4.

$\delta^{G_1} = 3$

G_2 :



is a pebbling of $T_{2,3}$.
(binary tree with 3 levels)

$\delta^{G_2} = 8$

Definition (Optimal Pebbling number)

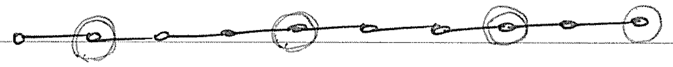
The optimal pebbling number of G, denoted by $f'(G)$, is $\min.\{\delta^G \mid \delta_G \text{ is a pebbling of } G\}$.

(*) $\delta^{C_4} = 3$, $\delta^{T_{2,3}} = 7$ (?)

Proposition $f'(G) \le 2\gamma(G)$.

Corollary $f'(P_n) \le 2 \cdot \lceil \frac{n}{3} \rceil$.

Example $n=10$

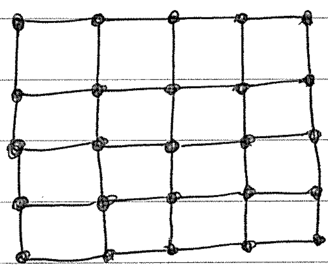


$f'(P_{10}) \le 8$, ($f'(P_{10}) \le 7$ (?))

Remark 1 Finding the upper bound of $f'(G)$ is comparatively easier than find a "good" lower bound.

Remark 2 The optimal pebbling number provides a better idea in domination; some vertex with stronger power can dominate some vertices which are far away from the vertex, not only adjacent vertices.

Good problem to work:

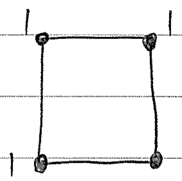


$f'(P_5 \square P_5) = ?$

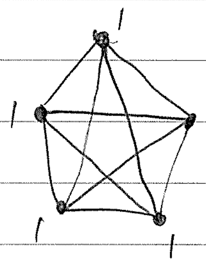
<u>Definition</u> (Pebbling number)

The pebbling number of a graph G, denoted by $f(G)$, is the minimum number of pebbles "t" such that any distribution configuration of these t pebbles, we have a pebbling of G.

<u>Example</u> $f(C_4) > 3$ and $f(K_n) > n-1$.



Not a pebbling



$f(K_5) > 4$.

<u>Example</u> $f(P_{10}) > 2^9 - 1$. (?)

<u>Remark</u> To determine $f(G)$ for general graph G is also a very difficult problem.

<u>Remark</u> This is a "worst case works" pebbling.