(König)

<u>Theorem</u> If G is a bipartite r-regular graph, then G contains r edge-disjoint 1-factors.

<u>Proof.</u> It suffices to prove that G contains a 1-factor. (Then, we have an $(r-1)$-regular graph left after taking away the 1-factor.)

Let $G=(A,B)$. Then, $|A|=|B|=n$. Let $A=\{a_1,a_2,\cdots,a_n\}$. Consider any $1\le k\le n$, $\bigcup_{j=1}^{k} N(a_{i_j}) = B' \subseteq B$. Since G is r-regular, the total number of edges incident to B' is $k\cdot r$. By the fact that each vertex in B is also of degree r, $|B'| \ge \frac{k\cdot r}{r} = k$. This concludes the proof by using Hall's condition. ■

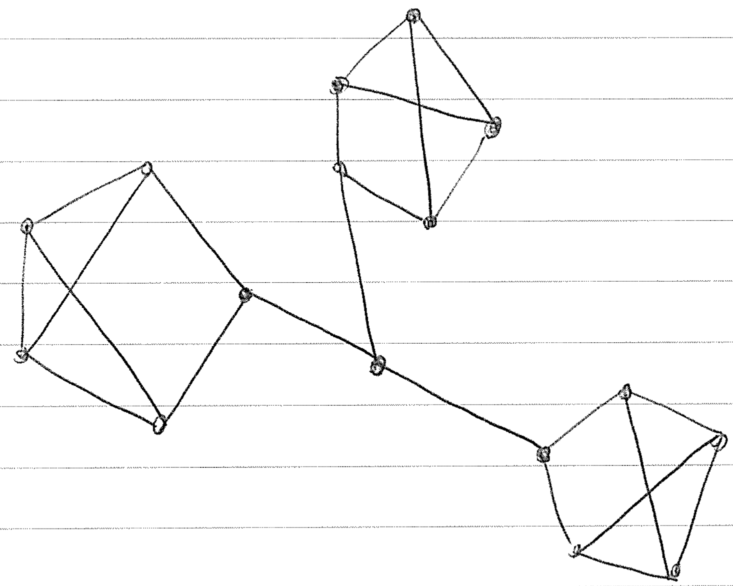
<u>Corollary</u> If G is a bipartite graph, then $\chi'(G)=\Delta(G)$.

<u>Note</u> If G is a bipartite graph, then there exists a bipartite graph $\tilde{G} \ge G$ such that $\tilde{G}$ is $\Delta(G)$-regular.

1-factor, 2-factors

Review

A k-factor of a graph G is a k-regular spanning subgraph of G. Clearly, if G is a k-regular graph, then G itself is a k-factor of G. The following theorem provides a necessary and sufficient condition for the existence of a 1-factor. First, we look at an example.



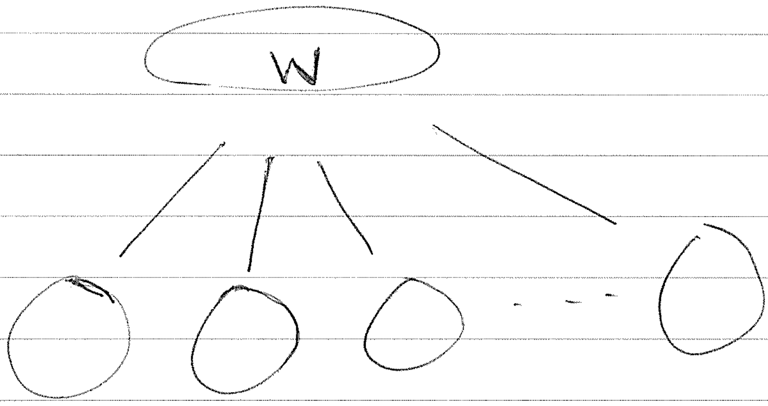
Can you find a 1-factor in this graph?

(*) If G has a 1-factor, then |G| must be even.

Theorem (Tutte, 1954)

A nontrivial graph G has a 1-factor if and only if for every proper subset S of $V(G)$, the number of odd components of $G-S$ does not exceed $|S|$.

Proof. $(\Rightarrow)$



#(odd components) $= t$, t can not be larger than $|W|$.

$(\Leftarrow)$ Omitted here.

Theorem (Petersen)

Every bridgeless cubic graph can be expressed as the edge sum of a 1-factor and a 2-factor.

Proof. It suffices to prove such a graph does contain a 1-factor.

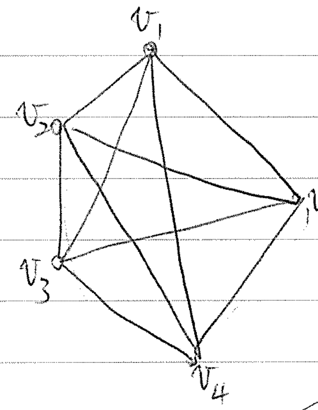
Suppose not. (G contains no 1-factors.) Let S be a set of vertices such that $|S| = k$ and $G-S$ contains n $(>k)$ odd components $G_1, G_2, \cdots, G_n$. Note that each G_i, $i=1,2,\cdots,n$, must be joined to S by at least one edge, for otherwise, G_i itself is a cubic graph of odd order. In fact, G_i is joined to S by at least two edges, since G contains no bridges. Moreover, if G_i joins S by exactly two edges, then G_i contains an odd number of odd vertices which is not possible. Hence each G_i joins S with at least three edges, and thus there are at least $3n$ edges join $\bigcup_{i=1}^{n} V(G_i)$ to S. But, $|S| = k$ and G is cubic graph, S has at most $3k$ edges joining the vertices outside of S. This is not possible. ∎

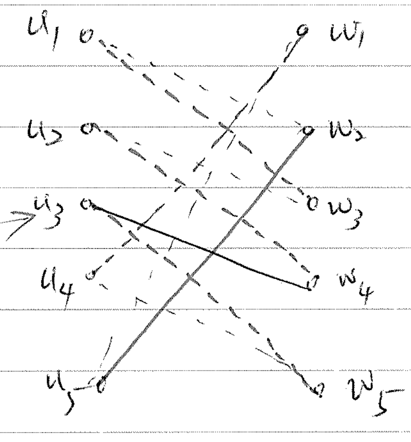
Theorem (Petersen)

If G is a $2n$-regular graph, then G contains n edge-disjoint 2-factors.

Proof. Clearly, G contains an eulerian circuit C. Let $V(G) = \{v_1, v_2, \cdots, v_p\}$. Construct a bipartite graph $H = (A, B)$ by way of C. Set $A = \{u_1, u_2, \cdots, u_p\}$, $B = \{w_1, w_2, \cdots, w_p\}$ and $E(H) = \{u_i w_j \mid v_j$ follows (immediately) v_i in $C\}$. Now, H is an n-regular graph. Therefore, H contains n 1-factors. Since each 1-factor is corresponding to a 2-factor in G, we have the proof. ∎

(Note: You can consider a 1-factor in H as a permutation of $\{v_1, v_2, \cdots, v_p\}$, see it?)





$C = (v_1, v_2, v_3, v_4, v_5, v_1, v_3, v_5, v_2, v_4)$

<u>Theorem</u> Every 2-connected cubic planar graphs G can be decomposed into three 1-factors.

<u>Proof</u>. It suffices to prove that $\chi'(G)=3$. Clearly, $\chi'(G)\geq 3$ since G is 3-regular. Now, we give a 3-edge-coloring of G. By 4-color theorem, G has a map coloring φ which uses four colors. Let the set of colors be $\mathbb{Z}_2\times\mathbb{Z}_2=\{(0,1),(0,0),(1,1),(1,0)\}$. By the fact that G is 2-connected, every edge e is on the boundary of exactly two regions R_1 and R_2. Hence, we can color e by the sum of the colors of R_1 and R_2. By the fact that these two colors are different, the color of each edge is in the set $\{(0,1),(1,0),(1,1)\}$, i.e., we use only three colors. So, we check the coloring we use is a proper coloring in what follows.

Let $|e\cap f|=1$. Then, e and f are two incident edges of a region R. Since G is 3-regular, e and f are also in two incident regions of R_1 and R_2. Now, the proof follows by the fact $\varphi(R)+\varphi(R_1)\neq\varphi(R)+\varphi(R_2)$.