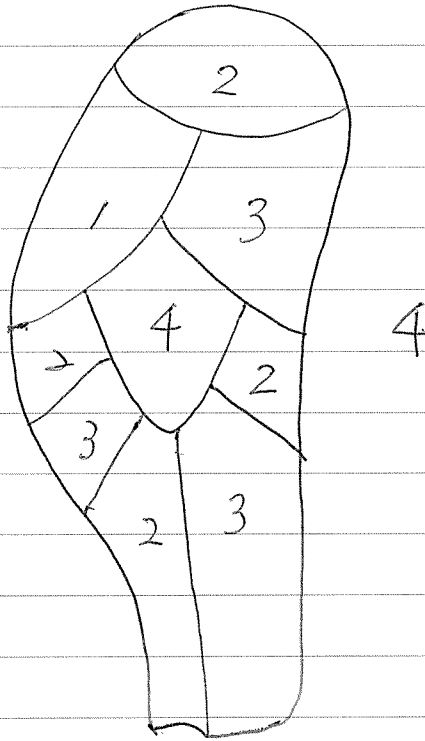


Graph Coloring

Week 6, Oct. 25, 27

NO. 3-1
DATE



Map Coloring Problem

1852, Oct. 23, Augustus De Morgan ^{sent a letter} → William Rowan Hamilton
(University College, London) (Trinity College, Dublin)

... My pupil says he guessed 4 colors are enough to

color the compartments differently so that figures with any
(of England)
portion of common boundary are differently colored. ...

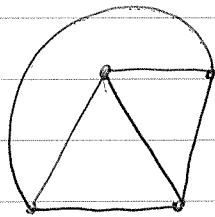
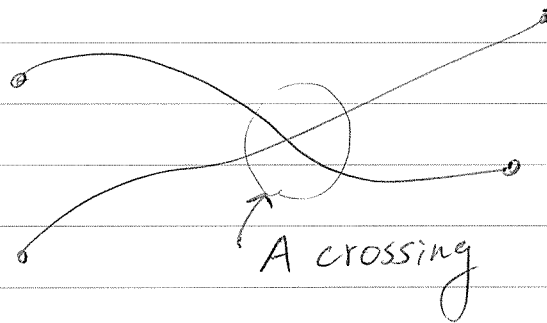
Student : Frederick Guthrie

Learned this idea from his brother : Francis Guthrie
(He provided an incomplete proof.)

Definition (Region Colorable)

A plane graph G is said to be n -region colorable if the regions of G can be colored with n or fewer colors so that adjacent regions are colored differently.

Note A plane graph G is a graph which can ^{be} drawn on a plane such that no two edges have a crossing.



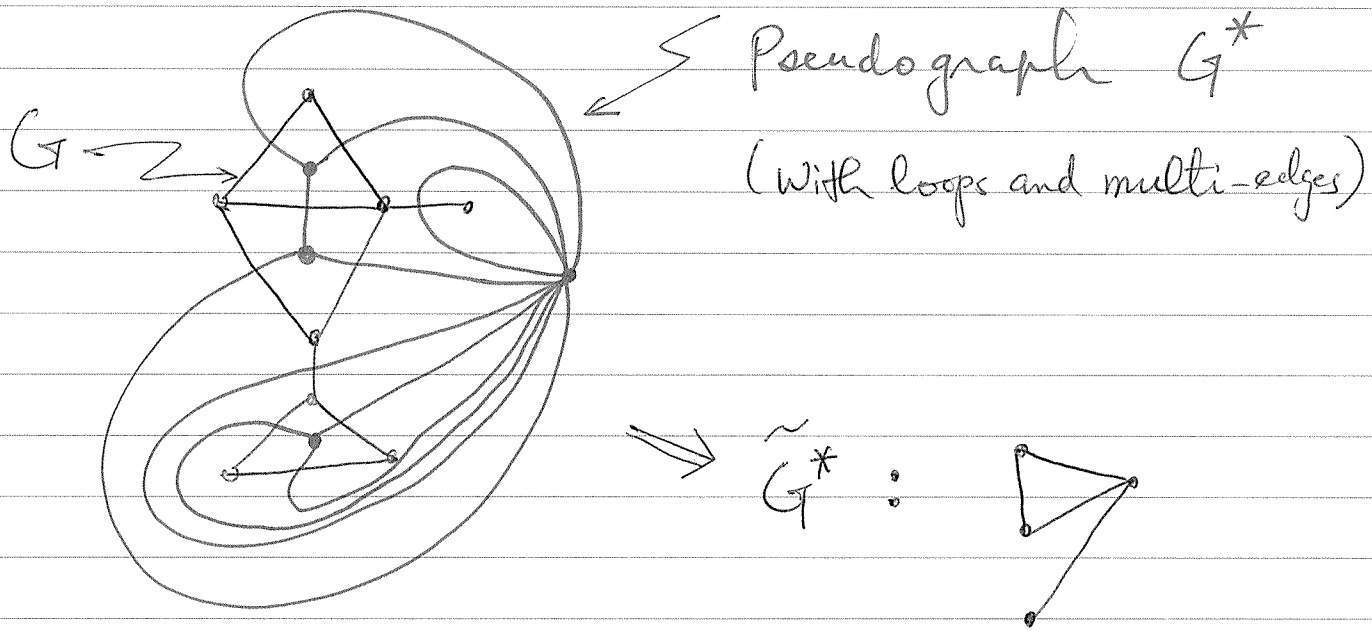
K_4 is a plane graph, but not K_5 .

Remark We shall talk about crossings of graphs later.

The Four Color Conjecture

Every map (plane graph) is 4-region colorable.

Now, it is known as the 4-color Theorem, 4CT in short.



Step 1 For each region of a plane graph, define a vertex.

Step 2 Join two vertices with an edge (or loop) if they have a common boundary.

Step 3 This pseudograph is known as the dual of the plane graph (G^*) .

Fact $(G^*)^* \cong G$.

Fact \tilde{G}^* is called the underlying graph of G^* if loops are deleted and replace the multiple edges by a single edge.

So, we can use the vertex coloring of \tilde{G}^* to color the regions of a plane graph G .

Definition (Vertex coloring)

A vertex k -coloring φ is a mapping $\varphi: V(G) \rightarrow [1, k]$ such
(k -coloring in short)

that $\varphi(u) \neq \varphi(v)$ if uv is an edge of G . The minimum

number k_x is called the chromatic number of G denoted by
such that G has a k -coloring

$\chi(G)$. (If G has a k -coloring, G is said to be k -colorable.)

(Fact 1) If $G \cong K_k$, then $\chi(G) \geq k$.

(Fact 2) If G is a bipartite graph, then $\chi(G) = 2$ provided
 G contains at least one edge.

(Fact 3) If G has a k -coloring $\varphi: V(G) \rightarrow \{1, 2, \dots, k\}$, then
for each $c \in [1, k]$, $\varphi^{-1}(c) \subseteq V(G)$ is an independent
set.

(Fact 4) $\chi(K_n) = n$ and $\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even;} \text{ and} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$

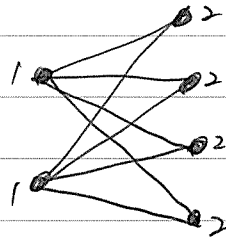
(Fact 5) If $h \geq k$ and G is k -colorable, then G is also h -colorable.

Remark 1 If G has a k -coloring φ such that

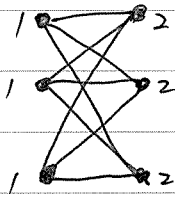
$$||\varphi^{-1}(c)| - |\varphi^{-1}(d)|| \leq 1 \text{ for any } c, d \in [1, k], \text{ then } \varphi$$

is an equitable k -coloring.

Remark 2 If G has a k -coloring, then G "may not" have an equitable k -coloring or equitable $(k+1)$ -coloring.



$$\chi(K_{2,4}) = 2$$



$$\chi(K_{3,3}) = 2$$

No equitable 3-coloring of

$K_{3,3}$ exists. (?)

Definition (critically n -chromatic)

For each $n \geq 2$, G is critically n -chromatic if $\chi(G) = n$

and $\chi(G - v) = n - 1$ for all $v \in V(G)$.

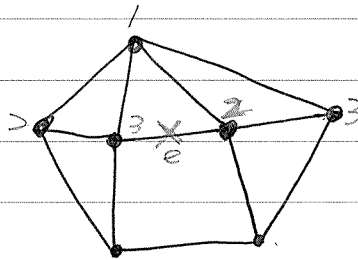
Example K_n is critically n -chromatic.

C_{2k+1} is critically 3-chromatic.

Definition (Minimally n -chromatic)

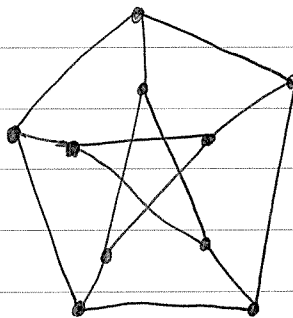
A graph G is minimally n -chromatic if $\chi(G) = n$ and

$\chi(G-e) = n-1$ for all edges $e \in E(G)$.



Critically 4-chromatic

Not minimally
4-chromatic
 $\chi(G-e) = 4$



Not critically 3-chromatic

Not
minimally
3-chromatic

Theorem (Brooks) 1941

If G is a connected graph that is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta(G)$.

Proof. (Omit here.)

(*) It is easier to consider the graphs which are not regular.
connected

Theorem (Halim, ¹⁹⁶⁷ Szekeres and Wilf)

For every graph G , $\chi(G) \leq 1 + \max \delta(G')$, where the maximum is taken over all induced subgraph G' of G . (Notice that if G is not a regular graph, then $\chi(G) \leq 1 + \delta(G) \leq \Delta(G)$.)

Proof. First, we claim that if G is critically n -chromatic, then $\delta(G) \geq n-1$. Suppose not. Let $\delta(G) \leq n-2$ and $\deg(v) = \delta(G)$ where $v \in V(G)$. By assumption, $\chi(G-v) = n-1$. Now, G has an $(n-1)$ -coloring since v can be colored with the color missing in its neighbors at most $(n-2)$ vertices. This is a contradiction to $\chi(G) = n$.

Now, we are ready for this theorem. Let H be an induced subgraph of G which is n -critical. (We can find this graph by

deleting vertices if necessary. So, $\delta(H) \leq \max_{G' \leq G} \delta(G')$. By

the fact that $\delta(H) \geq n-1$, $\max_{G' \leq G} \delta(G') \geq n-1 = \chi(G) - 1$.

Hence $\chi(G) \leq 1 + \max_{G' \leq G} \delta(G')$. ■

Theorem (Gallai, 1968)

For any graph G , $\chi(G) \leq 1 + m(G)$, where $m(G)$ denotes the length of a longest path in G . (If G does have a Hamilton path, then $\chi(G)$ is smaller.)

Proof. If H is an n -critical induced subgraph of G , then $\delta(H) \geq n-1$ and therefore H contains a path of length $n-1$.

This implies that $m(G) \geq n-1 = \chi(G) - 1$ and $\chi(G) \leq m(G) + 1$. ■

Greedy Coloring Algorithm

Let $\{1, 2, \dots, k\}$ be the colors we plan to use. Color a vertex with smaller integer if it is available.

