# Euler Circuit and Hamiltonian

1. Walk, Trial and Internal vertex

   - A **walk** in a graph $G$ is an alternating sequence of vertices and edges,

     $$W = v_0, e_1, v_1, e_1, \ldots, e_n, v_n,$$

     such that for $j = 1, \ldots, n$, the vertices $v_{j-1}$ and $v_j$ are the endpoints of the edge $e_j$.
   - A **trail** in a graph is a walk such that no edge occurs more than once.
   - A **u-v walk** is defined as a sequence of vertices starting at u and ending at v, where consecutive vertices in the sequence are adjacent vertices in the graph.
   - A **u-v trail** is a u-v walk, where no edge is repeated (each edge is used at most once).
   - A **u-v path** is a u-v walk, where no vertex is repeated (each vertex is used at most once).
   - An **internal vertex** is a vertex of degree at least 2.

2. Path and Cycle

   - A **path** in a graph is a trail such that no internal vertex is repeated.
   - A **cycle** is a closed path of length at least 1.

3. Eulerian graph

   A graph is **Eulerian** if it has a closed walk that contains every edges exactly once. **Eulerian circuit** or **Euler tour** in an undirected graph is a cycle that uses each edge exactly once.

4. De Bruijn Sequence

   A cylic sequence $(a_1, a_2, ..., a_{2^n})$ is $(2, n)-$de Bruijn sequence if it satisfy the following conditions:

   (a) $a_i \in \{0, 1\}, i = 1, 2, ..., 2^n$; and

   (b) $(a_j, a_{j+1}, ..., a_{j+n-1}), j = 1, 2, ..., 2^n$, (mod $2^n$), is $2^n$ difference $n-$dimensions.

5. $(2, n)$-De Bruijn digraph

   A $(2, n)$**-De Bruijn digraph** $D_{2,n}$ is a weighted digraph satisfy

(a) $V(D_{2,n}) = (\mathbb{Z}_2)^{n-1}$; and

(b) From $(a_1, a_2, \ldots, a_{n-1})$ to $(a_2, a_3, \ldots, a_n)$ is an arc with weight $(a_1, a_2, \ldots, a_n)$.

6. Postman tour

   A **postman tour**(or **covering walk**) is a closed directed walk that uses each arc at least once.

7. Optimal postman tour

   An **optimal postman tour** in a weighted graph, is a postman tour such that the sum of the weights of all the arcs is the minimum.

8. Hamiltonian Path, Hamiltonian Cycle and Hamiltonian graph

   - a **Hamiltonian path** (or traceable path) is a path in an undirected or directed graph that visits each vertex exactly once.
   - A **Hamiltonian cycle** (or Hamiltonian circuit) is a Hamiltonian path that is a cycle.
   - A graph that contains a Hamiltonian cycle is called a **Hamiltonian graph**.

9. Closure

   The (Hamiltonian) **closure** $C(G)$ of a graph $G$ is the graph resulting from adding edges between non-adjacent vertices of degree sum $|G|$ until it is impossible to do so any further.

10. Independent set and Independence Number

    $G = (V, E)$ is a graph. $S \subseteq V$.

    - If no two distinct vertices in $S$ of which are adjacent, then $S$ is an **independent set** or **stable set**.
    - If $S^*$ is a maximum independent set for $G$, then $\mid S^* \mid$ is called **independence number** of $G$, written $\beta(G)$

11. Power of Graph

    The $k$**-th power** $G^k$ of an undirected graph $G$ is another graph that has the same set of vertices, but in which two vertices are adjacent when their distance in G is at most $k$.

i.e. $E(G^k) = \{(u, v)|u, v \in V(G), 1 \leq d_G(u, v) \leq k\}$.

12. Difference

    For a vertex set $\mathbb{Z}_n$, the **difference** of $i$ and $j$ is $d(i, j) = \min\{|j - i|, n - |j - i|\}$.

13. Girth

    The **girth** of a graph with a cycle is the length of its shortest cycle.

    An acyclic graph has infinite girth.

# Connected

14. Connectivity

    The **connectivity** of a graph $G$, denoted $\kappa(G)$, is defined as

    $\kappa(G) = \min\{|S| | S \subseteq V(G)$ such that $G - S$ is a disconnected or has only one vertex$\}$.

15. Cut-edge and Cut-vertex

    A **cut-edge** (or **cut-vertex**) is an edge (or a vertex) whose removal increases the number of components.

16. Separating set and Vertex cut

    - A **separating set** or vertex cut of a graph $G$ is a set $S \subseteq V(G)$ such that $G - S$ has more than one component.
    - A graph $G$ is $k$**-connected** if its connectivity is at least $k$.

17. Edge-connectivity and $k$-edge-connected

    - The **edge-connectivity** of a connected graph $G$, denoted $\kappa'(G)$, is the minimum size of an edge subset $F$ such that $G - F$ is not connected.
    - $G$ is $k$**-edge-connected** $\kappa'(G) \geq k$.

18. Separating set, Vertex cut, Connectivity, and k-connected

    - The **separating set** or **vertex cut** of a digraph $D$ is a subset $S \subseteq V(D)$ such that $D - S$ is not strongly connected.
    - The **connectivity** of digraph $D$ is
      $\kappa(D) = \min\{|S| | S \subseteq V(G), D - S$ is not strongly connected or contains only one vertex$\}$
    - $D$ is a $k$-connected if the connectivity of $D$ is at least $k$.

19. Edge cut

    An **edge cut** in a strongly connected digraph $G = (V, E)$ is an arc subset $F \subset E$ such that the arc-deletion subdigraph $G - F$ is not strongly connected.

20. Network, Flow, and Feasible flow

- A **network** is a digraph satisfying the following conditions:
  - A **source vertex** $s$ and a **sink vertex** $t$ are in the network.
  - A nonnegative **capacity function** $cap : E \to \mathbb{N}$ exists.
- Let $f : E(D) \to \mathbb{R}$ be given for a digraph $D$. The function $f$ is called a **flow** if for every $v \in V(D)$, $\Sigma_{a \in E_v^+} f(a) = \Sigma_{a \in E_v^-} f(a)$.
- A **feasible flow** is a function $f : E \to \mathbb{R}$ which obeys the following constraints:
  - capacity constraints: $f(v, w) \leq \text{cap}(v, w)$, for each arc $(v, w) \in E$.
  - conservation constraints: $\Sigma_{(w,v)\in E} f(w, v) = \Sigma_{(v,w)\in E} f(v, w)$ for each vertex $v \in V - \{s, t\}$
  - nonnegativity constrains: $f(v, w) \geq 0$, for each arc $(v, w) \in E$.
- Let $e$ be an arc in a network, then $e$ is **saturated** if $f(e) = cap(e)$, and $e$ is **unsaturated** if $f(e) < cap(e)$.

21. Semipath, $f$-augmenting path, and Tolerance

Let a feasible flow $f$ in a network $N$, and $C$ is the capacity function.

- Any path in the underlying graph of $N$ in $G$ is called the **semipath** of $N$.
- Let $P$ be a semipath which starts from source vertex $s$ to sink vertex $t$, and for any $e \in E(P)$, if $P$ is an $f$-**augmenting path** then the following two statements are true.
  - If the directions of $P$ and arc $e$ are respectively the same, then $f(e) < C(e)$.
  - If the directions of $P$ and arc $e$ are respectively different, then $f(e) > 0$.
- $P$ is an $f$-augmenting path, the function $\epsilon$ is defined by the following:

$$\epsilon(e) = \begin{cases} C(e) - f(e), & \text{if the direction of } P \text{ and arc } e \text{ are the same.} \\ f(e), & \text{otherwise.} \end{cases}$$

  Then $min_{e \in E(P)} \epsilon(e)$ is the **tolerance** of $P$.

22. Net flow

Let $f$ be a flow in a network $N$.

- Let $x \in V(N)$,

  $f^+(x) - f^-(x)$ is called **net flow leaving** $x$.

  $f^-(x) - f^+(x)$ is called **net flow into** $x$.

- The net flow into sink vertex $t$ is called the **value** of $f$, denoted $val(f)$.

23. Maximum flow

    The maximum feasible flow $f$ is called **maximum flow**, and $val(f)$ is called maximum flow value.

24. Zero flow

    A flow $f$ in a network $N$ is called **zero flow** if $f(e) = 0$, $\forall e \in E(N)$.

25. Source set, Sink set, Source/Sink cut, and Capacity

    Let $s$ be the source vertex, $t$ be the sink vertex in a network $N$, and $S$ and $T$ be a partition of vertex set $V(N)$ which satisfy $s \in S$, $t \in T$, then $S$ is a **source set**, and $T$ is a **sink set**. Let $[S,T]$ be the set of arcs whose tails in $S$ and heads in $T$. $[S,T]$ is called the **source/sink cut** in network $N$. If $C$ is the capacity function of $N$, then the **capacity** of $[S,T]$ is defined by $cap(S,T) = \Sigma_{e \in [S,T]} C(e)$.

# Graph Coloring

26. Vertex labeling, $k$-coloring, Colors, Color class, Proper, $k$-colorable, and Chromatic number

    - A **vertex labeling** is vertex with labels in a graph $G$, typically $v_1, v_2, \ldots, v_n$ assigned to the vertices. $v_i$ is the label for $i = 1, 2, \ldots, n$.
    - A $k$-**coloring** of a graph $G$ is a function from its vertex-set $V(G)$ vertices to a set $C = \{1, 2, \ldots, k\}$ whose elements are called **colors**.
    - A **color class** for a graph with a coloring is the set of all vertices that are assigned the same color.
    - A $k$-coloring is **proper** if two adjacent vertices are always assigned different colors.
    - A graph is $k$-**colorable** if it has a proper $k$-coloring with $k$ or fewer colors.
    - The **chromatic number** of a graph $G$, denoted $\chi(G)$, is the smallest number $k$ of colors such that $G$ is $k$-colorable.

27. Proper subgraph

    $H$ is a **proper subgraph** of graph $G$ if $H \lneq G$.

28. $k$-chromatic, Optimal coloring, Color-critical, and $k$-critical

    - $G$ is a $k$-**chromatic** if $\chi(G) = k$.
    - $G$ is a **color-critical** or $k$-**critical** if $\chi(G) = k$ and for any $H$ which is the proper subgraph of $G$, $\chi(H) < k$.

29. Outerplanar Graph

    An undirected graph is an **outerplanar graph** if it can be drawn in the plane without crossings in such a way that all of the vertices belong to the unbounded face of the drawing.

30. Perfect

    A graph is **perfect** if every induced subgraph has chromatic number equal to the size of maximum clique.

31. Intersection graph

    For each integer $p \geq 1$, the $p$-**intersection graph** of a family $\mathscr{F} = \{S_1, \ldots, S_n\}$ of subsets of a finite set $S$ is defined to be the graph $G$ having $V(G) = \mathscr{F}$ with $S_iS_j \in E(G)$ if and only if $i \neq j$ and $|S_i \cap S_j| \geq p$.

32. Intersection graph and Interval graph

    - The **intersection graph** $G(\mathscr{F})$ of a family $\mathscr{F}$ of subsets of a given set has as its vertices the members of $\mathscr{F}$; two vertices are adjacent if and only if the corresponding subsets of $\mathscr{F}$ have non-empty intersection.
    - A graph $G$ is an **interval graph** if it is isomorphic to the intersection graph of a family of intervals of a line.

33. Chordal graph

    A **chordal graph** is one in which all cycles of four or more vertices have a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle.

34. Subdivision

    A **subdivision** of edge $e = uv$ in graph $G$ is the graph obtained from $G$ by replaying $e$ by the path $< u, w, v >$ where $w$ is a new vertex of degree two.

35. $H$-subdivision

    The graph $H'$ is a subdivision of $H$, if one can obtain $H'$ from $H$ by a series of edge subdivisions.

36. Chromatic Polynomial

    The **chromatic polynomial** $P(G, \lambda)$, $\lambda \in \mathbb{N}$, of graph $G = (V, E)$ is the function whose value at $\lambda$ ($\lambda = 1, 2, 3, \ldots$) is the number of proper colorings $\varphi : V \to \{1, \ldots, \lambda\}$ of $G$ with at most $\lambda$ colors. Here, two colorings are counted as different even if they yield the same color class by renumbering the colors.

37. Overfull

    An **overfull** graph $G$ is a graph whose size is greater than the product of its maximum degree and half of its order floored, i.e. $|E(G)| > \Delta(G)\lfloor\frac{|V(G)|}{2}\rfloor$.

38. Class 1 and Class 2

- A simple graph $G$ is **Class** 1 if $\chi'(G) = \Delta(G)$.
- A simple graph $G$ is **Class** 2 if $\chi'(G) = \Delta(G) + 1$.

39. Matching, M-saturated and Perfect matching

- A **matching** in $G = (V, E)$ is a set $M \subseteq E$ of pairwise nonadjacent edges.
- A vertex $v$ is **M-saturated** if $v$ is incident to an edge of $M$.
- A **perfect matching** in $G = (V, E)$ is a matching $M$ in which each vertex of $V$ is incident on exactly one edge of $M$.

40. Maximal matching and Maximum matching

- $M$ is a **maximal matching** if it is not a proper subset of any other matching in graph $G$. In other words, a matching $M$ of a graph $G$ is a maximal matching if every edge in $G$ has a non-empty intersection with at least one edge in $M$.
- A **maximum matching** of $G$ is a matching $M$ having the largest size.

41. Total coloring, Total chromatic number, $k$-total coloring and $k$-total colorable

- **Total coloring** is a type of graph coloring on the vertices and edges of a graph. When used without any qualification, a total coloring is always assumed to be proper in the sense that no adjacent edges and no edge and its endvertices are assigned the same color.
- The **total chromatic number** $\chi''(G)$ of a graph G is the least number of colors needed in any total coloring of G.
- A graph $G$ is called $k$**-total coloring** if the total coloring number is $k$.
- A graph $G$ is called $k$**-total colorable** if $G$ has a $k$-total coloring.

42. Type I and Type II

    $G$ is called **Type I** if $\chi''(G) = \Delta(G) + 1$, otherwise $G$ is called **Type II**.

43. Biconformability

Given an equibipartite graph $G$ and given a vertex-colouring which assigns the colours $c_1, c_2, \ldots, c_{\Delta(G)+1}$, let $A_i$ be the set of vertices of $A$ coloured $c_i$ and $B_i$ the set of vertices of $B$ coloured $c_i$. Let $a_i = |A|$ and $b_i = |B|$. If $W$ is a subset of $V(G)$, let $V_{<\Delta}(W)$ denote the set of vertices in $W$ which have degree less than $\Delta$ in the graph $G$. Call $G$ **biconformable** if $G$ has a vertex-colouring such that for $1 \leq i \leq \Delta(G)+1$,

$$|V_{<\Delta}(A \setminus A_i)| \geq b_i - a_i,$$

$$|V_{<\Delta}(B \setminus B_i)| \geq a_i - b_i,$$

and

$$\mathrm{def}(G) \geq \sum_{i=1}^{\Delta(G)+1} |a_i - b_i|.$$

44. List coloring and Choice number

- A (vertex) **list assignment** $L$ on a graph $G$ associates a set $L_v$ of colors with each vertex $v$ of $G$. Each $L_v$ is interpreted as the set of followed colors for vertex $v$.
- The graph $G$ is **$L$-colorable** (or **list colorable**, when $L$ is understood from context) if it admits a proper vertex-coloring $\varphi$ such that $\varphi(v) \in L_v$ for all $v$.
- If $|L_v| = k$ for all $v \in V$, then the list assignment $L$ is called a $k-$**assignment** $L$.
- The **choice number** of $G$, denoted $ch(G)$, is the smallest nonnegative integer $k$ such that $G$ is $k-$choosable. (In part of the literature, the choice number is called $listchromaticnumber$, and also the notation $\chi_l(G)$ is commonly used for $ch(G)$.)

45. Edge choice number

The **edge choice number** (or **list chromatic index** or **list edge chromatic number**) is the minimum list-size that guarantees a list edge-coloring of $G$; it is denoted by $ch'G$ (or by $\chi'_l(G)$)

46. Edge labeling, $k$-edge-coloring, Proper, $k$-edge-colorable and Chromatic index

- An **edge labeling** is a function mapping $E$ to a set of labels.

- A $k$**-edge-coloring** is a function mapping $E$ to a $k$-set $\{1, 2, \ldots, k\}$ of labels. The labels $1, 2, \ldots, k$ is called **color**. A subset assigned to the same color is called a **color class**.
- A $k$-edge-coloring of a graph is almost always a **proper** coloring, namely a labelling of the edges with colors such that no two edges sharing the same vertex have the same color.
- A graph that can be assigned a (proper) $k$-edge-coloring is $k$**-edge-colorable**
- The smallest number of colors needed for an edge-coloring of a graph $G$ is the **chromatic index**, or edge chromatic number, $\chi'(G)$.