(4, 13)

Definition (Non-adaptive)

A group testing algorithm is non-adaptive if all tests must be specified without knowing the outcomes of other tests.

(Fact)

A nonadaptive algorithm can be represented by a $t \times n$ (0,1)-matrix if t tests are specified for a group testing on n items. (Matrix-representation)

Let M be the (0,1)-matrix, R_i and C_j denote row i and column j. We can view R_i (C_j) as a set of column indices (row indices) corresponding to the 1-entries.

$$M : \begin{array}{c|c} & C_1 \; C_2 \; \cdots \; C_n \\ \begin{array}{c} R_1 \\ R_2 \\ \vdots \\ R_t \end{array} & \left[\begin{array}{c} (0,1)\text{-matrix} \\ \text{(Can be extended} \\ \text{to general matrices)} \end{array} \right]_{t \times n} \end{array}$$

Definitions

1. Let A and B are two sets in $[1,n]$. Then the Boolean sum of A and B, denoted $A \oplus B$, is the set of elements in A or B.

2. We shall call $A \oplus B$ as the union of A and B.

3. In each C_j (R_i) is the set of row indices x (Column indices y) such that $m_{x,j} = 1$ in M. $(m_{i,y} = 1)$

4. Each C_j can be viewed as a set (subset of $[1,t]$).

5. M is called d-separable ($\bar{d}$-separable) if the unions of d columns (up to d columns) are all distinct.

6. M will be called d-disjunct if the union of any d columns does not contain any other column.

Group Testing and Superimposed Codes

(Non-adaptive)

(*) A $t \times n$ d-separable ($\bar{d}$-separable or d-disjunct) matrix generates a nonadaptive (d, n) algorithm of group testing with t tests by associating the columns with items, the rows with tests, and interpreting a 1-entry in cell (i,j) $(m_{ij} = 1)$ as the containment (item j is in pool i), and the 0-entry a noncontainment of item j in test (pool) i.

(**) Consider the set of columns $C_1, C_2, \cdots, C_n$ as n subsets of $[1, t]$. (is a superimposed code) ($M = \{C_1, C_2, \cdots, C_n\}$) with length t, volume n, distance D and strength d if $\forall\ S \subseteq [1, n]$, with $|S| \le d$ and $i \in [1, n] \setminus S$,

$$\left| C_i - \bigcup_{j \in S} C_j \right| \ge D.$$

$D = 1, \Rightarrow d$- disjunct

Let $z = D - 1$.

A matrix ($t \times n$ incidence matrix) of a superimposed code M with distance D and strength d is called a (d, z)-disjunct matrix.

$(d, 0)$-disjunct $\approx$ d-disjunct.

How to decode? (nonadaptive algorithm)

Requirements:

(1) d-separable

The unions of any d columns $\left(\text{in } \binom{[1,n]}{d}\right)$ must have distinct outcomes.

Let S, S' be two distinct d-subset of $[1,n]$.

o $\bigcup_{x \in S} C_x \neq \bigcup_{y \in S'} C_y \Rightarrow$ The group testing has different outcomes!

(?)

(2) $\bar{d}$-separable

For $|S| \le d$, $|S'| \le d$,

$$\bigcup_{x \in S} C_x \neq \bigcup_{y \in S'} C_y .$$

For example, if $n = 20$, $d = 3$, then we need $\binom{20}{3} + \binom{20}{2} + \binom{20}{1} + \binom{20}{0}$ distinct outcome vectors.

This fact provides a lower bound of the number of tests. Let t be the # of tests. Then

$$2^t \ge \underbrace{\binom{20}{3} + \binom{20}{2} + \binom{20}{1} + \binom{20}{0}}_{} .$$

$\downarrow$

$3 \times 19 \times 20 \quad 19 \times 10 \quad 20 \quad 1$

$1140 + 190 + 20 + 1$

$= 1351$.

$t \ge 11$!

(3) d-disjunct $\Rightarrow$ $\bar{d}$-separable

Proof.

Suppose not. There exist two distinct subset of $[1, t]$, S and S' such that $1 \le |S| \le d$, $1 \le |S'| \le d$, and $\bigcup_{x \in S} C_x = \bigcup_{y \in S'} C_y$. Let $i \in S \setminus S'$.

Then $C_i \subseteq \bigcup_{x \in S} C_x = \bigcup_{y \in S'} C_y$. Therefore, M is not d-disjunct.

Disjunctness $\Rightarrow$ Easy decoding

$$\begin{bmatrix} \quad & & \quad \\ & & \\ & & \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_t \end{pmatrix}$$

outcome vector

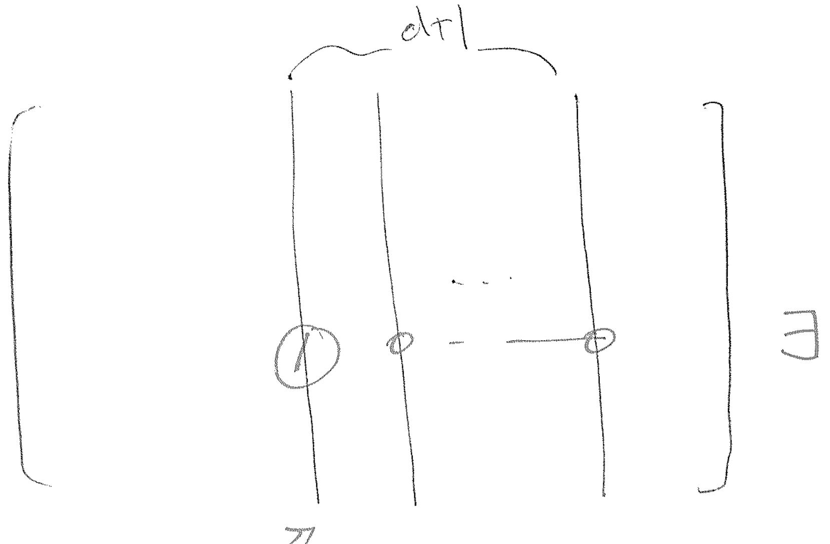
If $z_i = \begin{cases} 0, & \text{then } R_i \text{ is called a negative pool,} \\ 1, & \text{then } R_i \text{ ——— positive pool.} \end{cases}$

(***)
(Fact)

If $z_i = 0$, then $\forall\ x \in R_i$, the item corresponds to C_x is negative.

(Fact)

If M is d-disjunct, then the union of zero pools corresponds to at least $n-d$ pure (good) items.



If this item is negative, then it looks like |. Not possible!