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Prime labelling of graphs

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Definition and Preliminaries

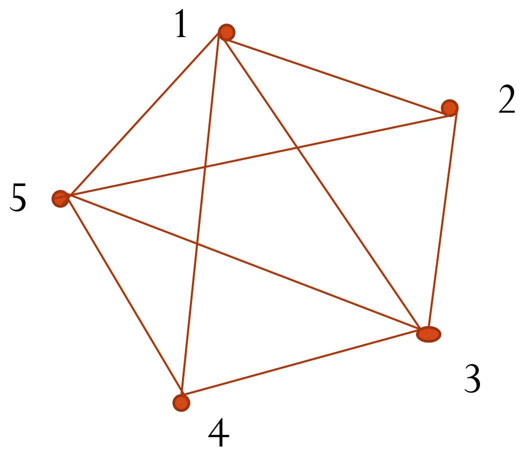
- Let G = (V, E) be a graph and n is the number of vertices in G. A *prime labelling* of the graph G is a bijection from V onto {1, 2, …, n} such that incident vertices receive coprime images.
- Example: Use 1, 2, …, n to label a cycle of order n, C_n, consecutively. Then, C_n has a prime labelling.
- Example: For $n \geq 4$, K_n does not have a prime labelling.
- It is not difficult to realize that the chance for a sparse graph to have a prime labelling is very high.

Tree conjecture

- Around 1980 Roger Entringer conjectured that every tree has a prime labelling.
- S. M. Lee et al. showed that several special classes of trees do have prime labelling such as caterpillar, star-like tree, ….
- K. C. Huang and myself verify the conjecture for smaller order (≤ 15) trees (DM, 1994).
- S. H. Lin (1999) further improved the above "small order" to 105.
- P. Haxcell et al. showed the conjecture is true for sufficiently large trees recently.
- Many other tries are not mentioned here! (Also other special graphs, see "A dynamic survey of graph labelling".)

Coprime graphs

- A coprime graph of order n is a graph whose vertex set is $\{1, 2, \ldots, n\}$ and two vertices are adjacent if and only if they are coprime.
- A coprime graph of order 5.



Verify the conjecture

- We can verify the tree conjecture by showing that any tree is isomorphic to a spanning subgraph (tree) of the coprime graph.
- If we plan to check if a graph does or does not have a prime labelling, this is another approach.
- We can easily see that an n-regular graph of order 2n does not have a prime labelling. (?)
- For each $x \in [1, n]$, let $r(x) = \#(\{y : y \in [1, n] \text{ and } g.c.d.(x, y) = 1\})$ be the rank of x in $[1, n]$. For example, the rank of 6 in $[1, 12]$ is 4 and the rank of 6 in $[1, 10]$ is 3. This implies that any 4-regular graph of order 10 does not have a prime labeling.

Observation

- Let G be a graph of order n with minimum degree $\delta(G)$. If $\delta(G)$ is larger than the minimum rank of integers in [1, n], then G has no prime labelling.
- Let P(n) denote the set of primes in [1, n] and $\pi(n)$ the number of primes in [1, n]. Let $\pi^+(n)$ denote the number of primes in (n/2, n]. Then a graph with a prime labelling can have at most $\pi^+(n) + 1$ vertices of degree n-1.
- The distribution of primes does play an important role in the study of prime labelling.

An idea of finding a prime labelling

- Let $\boldsymbol{S} = \{S_1, S_2, \ldots, S_t\}$ be a collection t subsets S. An ordered t-tuple $(b_1, b_2, \ldots, b_t)$ is called a system of distinct representatives (SDR) of $\boldsymbol{S}$ if for each $i \in \{1, 2, \ldots, t\}$, $b_i \in S_i$ and all b_i's are distinct.
- Theorem (P. Hall, 1935)

An SDR exists if and only if the union of any collection k sets from $\boldsymbol{S}$ contains at least k distinct elements.

- The above condition is known as Hall's condition.
- We shall apply this theorem to find a prime labelling for two classes of graphs: Certain Trees and prescribed 4-regular graphs.

About trees

- Let T be a tree of order n. (Bipartite graph)
- Let $T = (A, B)$ such that $A = \{a_1, a_2, \ldots, a_s\}$, $B = \{b_1, b_2, \ldots, b_t\}$, $s \leq t$, $\deg(a_i) \geq \deg(a_{i+1})$ and $\deg(b_j) \geq \deg(b_{j+1})$.
- Theorem If $s \leq \pi(n)$, then T has a prime labelling.

Proof. (Outline)

Step 1. Label the vertices of A by using 1 and odd primes starting from larger primes. For example, if $n = 100$, then label a_1 with 1, a_2 with 97, a_3 with 89, …, etc. In case that s is equal to $\pi(n)$, then a_s is labelled with 3.

Continued

Step 2. Let $L(A')$ be the set of labels which are used to label the vertices of $A' \subseteq A$. For each b_i, $i \in [1, t]$, define a set $S_i = \{x \in [1, n] / L(A): \text{g.c.d.}(x, y) = 1 \text{ for each } y \in L(N(b_i))\}$.

Step 3. Find an SDR for $\boldsymbol{S} = \{S_1, S_2, \ldots, S_t\}$ and the prime labeling can be defined accordingly.

Remark. If $|A| \leq \pi(n) - \pi(n/2) + 1$, then the proof follows directly from the fact that each S_i is of cardinality $n - s$.

Outline of proof

- For each $i \in [1, t]$, the cardinality of S_i is at least 1.
- For any two distinct i and j, the union of S_i and S_j has cardinality at least $\pi(n) - 1$.
- The union of $\pi(n) - 1$ S_i's has cardinality at least $n - \pi(n) - \lfloor n/3 \rfloor + 1$.
- The union of $n - \pi(n) - \lfloor n/3 \rfloor + 1$ S_i's has cardinality at least $n - \pi(n) - \lfloor n/15 \rfloor + 1$.
- The union of $n - \pi(n) - \lfloor n/15 \rfloor + 1$ S_i's has cardinality t.

Remark

- We are ***working*** on the case with larger cardinality for A which is at most half of n.
- The idea is to label the vertices of A with odd integers starting from 1, largest prime in [1, n], …, etc. following the decreasing order of degrees.
- In that case, the odd integers with lower rank will be used to label vertices with smaller degree in A.)
- Similarly, the proof follows by finding an SDR of a suitably defined collection of sets.

4-regular graphs with prime labelling

- Clearly, it is not true for all 4-regular graphs.
- For example, K_5 does not have a prime labelling. In fact, any 4-regular graph of order between 6 and 10 do not have a prime labelling since the rank of 6 is at most 3.
- Problem

1. Can we construct a 4-regular graph of order $n \geq 11$ which has a prime labelling? (All order!)

2. Prove that all connected 4-regular graphs of larger order do have a prime labelling. (Harder!)

Examples

- We can use two edge-disjoint Hamilton cycles to construct a 4-regular graph.
- The following two cycles are Hamilton cycles defined on the vertex set $\{v_1, v_2, \ldots, v_{12}\}$ and it is easy to check the union of these two cycles is a 4-regular graph:

  $(v_1, v_2, \ldots, v_{12})$ and $(v_2, v_9, v_4, v_1, v_6, v_{11}, v_8, v_3, v_{10}, v_7, v_{12}, v_5)$.
- Moreover, let $\phi(v_i) = i$ for $i = 1, 2, \ldots, 12$, we have a prime labelling of the above 4-regular graph.

Idea of construction

- We consider the case of even order here and the case when the order is odd can be obtained by a similar way.
- Let the order be 2m where $m \geq 6$.
- First, we draw a cycle of order 2m and their vertices are labelled with 1, 2, …, 2m respectively. (This is a cycle with a prime labelling.)
- We consider the above cycle is defined on the set [1,2m].
- Now we try to find a 2-factor defined on [1,2m] which is edge-disjoint with the above cycle and also the 2-factor has a prime labelling, moreover their union has a prime labelling.

The second cycle

- We use a Hamilton cycle for the 2-factor.
- The cycle represented by its labels $(a_1,a_2,\ldots,a_{2m})$ should satisfies the following conditions:

  1. g.c.d.$(a_i,a_{i+1}) = 1$ (i takes modulo 2m); and

  2. $| a_i - a_{i+1} | > 1$ (i takes modulo 2m).
- So, in the example for m = 12, the 2nd cycle is (5,2,9,4,1,6,11,8,3,10,7,12).
- Clearly, the construction of this cycle plays the key role!
- Again, we can apply the idea of SDR to construct this cycle.

  (?)

The construction of 2^{nd} cycle

- Let the cycle be denoted by $(x_1, x_2, \ldots, x_{2m})$, moreover, $x_{2i} = 2i$ for $i = 1, 2, \ldots, m$.
- Now, we need to find x_{2i-1} for $i = 1, 2, \ldots, m$.
- Let $S_i = \{y$: y is an odd integer, g.c.d.$(y, x_{2i-2}) = 1$, g.c.d.$(y, x_{2i}) = 1$ and y is not in $[2i-3, 2i+1]$ (i takes modulo 2m)$\}$.
- For example, if we try to find x_{17}, then x_{17} can't be odd integers 15, 17, 19 and also x_{17} must be coprime to both 16 and 18. Therefore, 1(if $m > 9$), 5, 7, 11, … are elements in S_{17}.
- An SDR of $\{S_1, S_2, \ldots, S_m\}$ gives the labelling of $x_1, x_3, \ldots, x_{2m-1}$ respectively and we have the cycle.

Thanks!