Prime labelling of graphs

Hung-Lin Fu
Department of Applied Mathematics
National Chiao Tung University
Hsin Chu, Taiwan

Work jointly with K. P. Chang and Jyhmin Kuo
Definition and Preliminaries

- Let $G = (V, E)$ be a graph and $n$ is the number of vertices in $G$. A *prime labelling* of the graph $G$ is a bijection from $V$ onto $\{1, 2, \ldots, n\}$ such that incident vertices receive coprime images.

- Example: Use 1, 2, \ldots, $n$ to label a cycle of order $n$, $C_n$, consecutively. Then, $C_n$ has a prime labelling.

- Example: For $n \geq 4$, $K_n$ does not have a prime labelling.

- It is not difficult to realize that the chance for a sparse graph to have a prime labelling is very high.
Tree conjecture

- Around 1980 Roger Entringer conjectured that every tree has a prime labelling.
- S. M. Lee et al. showed that several special classes of trees do have prime labelling such as caterpillar, star-like tree, ….
- K. C. Huang and myself verify the conjecture for smaller order (≤ 15) trees (DM, 1994).
- S. H. Lin (1999) further improved the above “small order” to 105.
- P. Haxcell et al. showed the conjecture is true for sufficiently large trees recently.
- Many other tries are not mentioned here! (Also other special graphs, see “A dynamic survey of graph labelling”.)
Coprime graphs

- A coprime graph of order $n$ is a graph whose vertex set is $\{1, 2, \ldots, n\}$ and two vertices are adjacent if and only if they are coprime.
- A coprime graph of order 5.
Verify the conjecture

• We can verify the tree conjecture by showing that any tree is isomorphic to a spanning subgraph (tree) of the coprime graph.
• If we plan to check if a graph does or does not have a prime labelling, this is another approach.
• We can easily see that an n-regular graph of order 2n does not have a prime labelling. (?
• For each $x \in [1, n]$, let $r(x) = \#(\{y : y \in [1, n] \text{ and } \gcd(x, y) = 1\})$ be the rank of $x$ in $[1, n]$. For example, the rank of 6 in $[1, 12]$ is 4 and the rank of 6 in $[1, 10]$ is 3. This implies that any 4-regular graph of order 10 does not have a prime labeling.
Observation

- Let $G$ be a graph of order $n$ with minimum degree $\delta(G)$. If $\delta(G)$ is larger than the minimum rank of integers in $[1, n]$, then $G$ has no prime labelling.

- Let $P(n)$ denote the set of primes in $[1, n]$ and $\pi(n)$ the number of primes in $[1, n]$. Let $\pi^+(n)$ denote the number of primes in $(n/2, n]$. Then a graph with a prime labelling can have at most $\pi^+(n) + 1$ vertices of degree $n-1$.

- The distribution of primes does play an important role in the study of prime labelling.
An idea of finding a prime labelling

- Let $S = \{S_1, S_2, \ldots, S_t\}$ be a collection of subsets $S$. An ordered $t$-tuple $(b_1, b_2, \ldots, b_t)$ is called a system of distinct representatives (SDR) of $S$ if for each $i \in \{1, 2, \ldots, t\}$, $b_i \in S_i$ and all $b_i$'s are distinct.

- **Theorem (P. Hall, 1935)**
  
  An SDR exists if and only if the union of any collection $k$ sets from $S$ contains at least $k$ distinct elements.

- The above condition is known as Hall’s condition.

- We shall apply this theorem to find a prime labelling for two classes of graphs: Certain Trees and prescribed 4-regular graphs.
About trees

- Let $T$ be a tree of order $n$. (Bipartite graph)
- Let $T = (A, B)$ such that $A = \{a_1, a_2, \ldots, a_s\}$, $B = \{b_1, b_2, \ldots, b_t\}$, $s \leq t$, $\deg(a_i) \geq \deg(a_{i+1})$ and $\deg(b_j) \geq \deg(b_{j+1})$.
- **Theorem** If $s \leq \pi(n)$, then $T$ has a prime labelling.

  **Proof.** (Outline)

  **Step 1.** Label the vertices of $A$ by using 1 and odd primes starting from larger primes. For example, if $n = 100$, then label $a_1$ with 1, $a_2$ with 97, $a_3$ with 89, \ldots, etc. In case that $s$ is equal to $\pi(n)$, then $a_s$ is labelled with 3.
Step 2. Let $L(A')$ be the set of labels which are used to label the vertices of $A' \subseteq A$. For each $b_i, i \in [1, t]$, define a set $S_i = \{x \in [1, n] / L(A): \gcd(x, y) = 1 \text{ for each } y \in L(N(b_i))\}$.

Step 3. Find an SDR for $S = \{S_1, S_2, \ldots, S_t\}$ and the prime labeling can be defined accordingly.

Remark. If $|A| \leq \pi(n) - \pi(n/2) + 1$, then the proof follows directly from the fact that each $S_i$ is of cardinality $n - s$. 

Continued
Outline of proof

• For each $i \in [1, t]$, the cardinality of $S_i$ is at least 1.
• For any two distinct $i$ and $j$, the union of $S_i$ and $S_j$ has cardinality at least $\pi(n) - 1$.
• The union of $\pi(n) - 1$ $S_i$’s has cardinality at least $n - \pi(n) - \lfloor n/3 \rfloor + 1$.
• The union of $n - \pi(n) - \lfloor n/3 \rfloor + 1$ $S_i$’s has cardinality at least $n - \pi(n) - \lfloor n/15 \rfloor + 1$.
• The union of $n - \pi(n) - \lfloor n/15 \rfloor + 1$ $S_i$’s has cardinality $t$. 
Remark

- We are *working* on the case with larger cardinality for $A$ which is at most half of $n$.
- The idea is to label the vertices of $A$ with odd integers starting from 1, largest prime in $[1, n]$, …, etc. following the decreasing order of degrees.
- In that case, the odd integers with lower rank will be used to label vertices with smaller degree in $A$.
- Similarly, the proof follows by finding an SDR of a suitably defined collection of sets.
4-regular graphs with prime labelling

- Clearly, it is not true for all 4-regular graphs.
- For example, $K_5$ does not have a prime labelling. In fact, any 4-regular graph of order between 6 and 10 do not have a prime labelling since the rank of 6 is at most 3.

**Problem**

1. Can we construct a 4-regular graph of order $n \geq 11$ which has a prime labelling? *(All order!)*

2. Prove that all connected 4-regular graphs of larger order do have a prime labelling. *(Harder!)*
Examples

• We can use two edge-disjoint Hamilton cycles to construct a 4-regular graph.

• The following two cycles are Hamilton cycles defined on the vertex set \( \{v_1, v_2, \ldots, v_{12}\} \) and it is easy to check the union of these two cycles is a 4-regular graph:
  
  \((v_1, v_2, \ldots, v_{12})\) and \((v_2, v_9, v_4, v_1, v_6, v_{11}, v_8, v_3, v_{10}, v_7, v_{12}, v_5)\).

• Moreover, let \( \phi(v_i) = i \) for \( i = 1, 2, \ldots, 12 \), we have a prime labelling of the above 4-regular graph.
Idea of construction

- We consider the case of even order here and the case when the order is odd can be obtained by a similar way.
- Let the order be $2m$ where $m \geq 6$.
- First, we draw a cycle of order $2m$ and their vertices are labelled with $1, 2, \ldots, 2m$ respectively. (This is a cycle with a prime labelling.)
- We consider the above cycle is defined on the set $[1,2m]$.
- Now we try to find a 2-factor defined on $[1,2m]$ which is edge-disjoint with the above cycle and also the 2-factor has a prime labelling, moreover their union has a prime labelling.
The second cycle

• We use a Hamilton cycle for the 2-factor.
• The cycle represented by its labels \((a_1, a_2, \ldots, a_{2m})\) should satisfy the following conditions:
  1. \(\gcd(a_i, a_{i+1}) = 1\) (i takes modulo 2m); and
  2. \(|a_i - a_{i+1}| > 1\) (i takes modulo 2m).
• So, in the example for \(m = 12\), the 2\textsuperscript{nd} cycle is 
  \((5, 2, 9, 4, 1, 6, 11, 8, 3, 10, 7, 12)\).
• Clearly, the construction of this cycle plays the key role!
• Again, we can apply the idea of SDR to construct this cycle.
The construction of 2\(\text{nd}\) cycle

- Let the cycle be denoted by \((x_1, x_2, \ldots, x_{2m})\), moreover, \(x_{2i} = 2i\) for \(i = 1, 2, \ldots, m\).
- Now, we need to find \(x_{2i-1}\) for \(i = 1, 2, \ldots, m\).
- Let \(S_i = \{y: y\) is an odd integer, \(\gcd(y, x_{2i-2}) = 1, \gcd(y, x_{2i}) = 1\) and \(y\) is not in \([2i-3, 2i+1]\) (\(i\) takes modulo 2\(m\))\}\).
- For example, if we try to find \(x_{17}\), then \(x_{17}\) can’t be odd integers 15, 17, 19 and also \(x_{17}\) must be coprime to both 16 and 18. Therefore, 1 (if \(m > 9\)), 5, 7, 11, … are elements in \(S_{17}\).
- An SDR of \(\{S_1, S_2, \ldots, S_m\}\) gives the labelling of \(x_1, x_3, \ldots, x_{2m-1}\) respectively and we have the cycle.
Thanks!