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Discharging Method

The so-called discharging method in graph theory is a technique used to prove the existence of certain structures hidden in a graph, mainly in planar graphs or the graphs which are 2-cell embedded in a surface.

So, it is used to prove that every graph in a certain class (say planar graphs) contains subgraphs from a special list. With the presence of the desired subgraph, we are able to prove some results which use the idea of reduction.

Step 1 A charge is assigned to each vertex and (or) each face of the graph. (Make sure the charges sum up to a finite positive integer with the aid of Euler's formula or Euler-Poincaré's formula.)

Step 2 Redistribute the charges to nearby vertices and (or) faces, as required by a set of discharging rules. (The is the discharging step.)

Step 3 We conclude from Step 2 that some vertices or 2 faces must have positive charges.

Step 4 Use the above step to conclude that certain subgraphs do exist.

Two examples

1. If a planar graph has minimum degree 5, then we have a light edge with endpoints of degrees 5, 5; or 5, 6.

Proof. Let $d(x)$ denote the degree (or length) of a vertex x (or face x).

Step 1. Charge each vertex v with $6-d(v)$ and each face f with $6-2d(f)$. So, the total charge is equal to 12.

Step 2. Discharge $\frac{1}{5}$ to each neighbor for each vertex of degree 5. Then, a vertex of degree v (finally) has a charge at most $6-d(v)+\frac{1}{5}\cdot d(v) = 6-\frac{4}{5}d(v)$.

Step 3. Only degree 5, 6, 7 vertices can have positive charges.

Step 4. If a degree 5, 6 vertex has positive charge, i.e., Then, they must have receive charges from nearby (adjacent) vertices, the proof follows.

On the other hand, if a vertex v of degree 7 has positive charge, then the vertex receive at least $6 \cdot \frac{1}{5}$ charges from 6 vertices which are incident v. Since, we consider a maximal planar graph, at least two of those vertices of degree 5 are incident. This concludes the proof.

2. Every 2-edge-connected C_3-free graph with min. degree at least 3 has either a C_4 containing a 3-vertex or a C_5 containing at least four 3-vertices.

Proof. Let G be the graph with $\delta(G) \geq 3$.

<u>Step 1</u> Charge each vertex v with $4 - d(v)$ and each face f with $4 - d(f)$. (The total charge is 8.)

<u>Step 2</u> Discharge every 3-vertex equally to its incident faces.

<u>Step 3</u> Every vertex has a non-positive charge and thus some face f must have a positive charge after discharging rule was applied.

<u>Step 4</u> Let k be the number of 3-vertices in f. Hence f receives $4 - d(f) + \frac{1}{3}k > 0$. Now, the conclusion follows.

$$d(f) \ge k$$

$$\Rightarrow 4 - d(f) + \frac{1}{3} d(f) > 0$$

$$\Rightarrow 4 > \frac{2}{3} d(f)$$

$$\Rightarrow 6 > d(f)$$

$$\Rightarrow 5 \ge d(f)$$

Since G has no C_3's, f must be either C_4 or C_5.

If the boundary of f is not a C_4 which contain a 3-vertex, then f is a face with boundary C_5 which has positive charge.

Then $4 - 5 + \frac{1}{3} k > 0$

$$\frac{1}{3} k > 1 \Rightarrow k > 3. \quad \blacksquare$$

Discharging method (discrete mathematics)

From Wikipedia, the free encyclopedia

The **discharging method** is a technique used to prove lemmas in structural graph theory. Discharging is most well known for its central role in the proof of the Four Color Theorem. The discharging method is used to prove that every graph in a certain class contains some subgraph from a specified list. The presence of the desired subgraph is then often used to prove a coloring result.

Most commonly, discharging is applied to planar graphs. Initially, a *charge* is assigned to each face and each vertex of the graph. The charges are assigned so that they sum to a small positive number. During the *Discharging Phase* the charge at each face or vertex may be redistributed to nearby faces and vertices, as required by a set of discharging rules. However, each discharging rule maintains the sum of the charges. The rules are designed so that after the discharging phase each face or vertex with positive charge lies in one of the desired subgraphs. Since the sum of the charges is positive, some face or vertex must have a positive charge. Many discharging arguments use one of a few standard initial charge functions (these are listed below). Successful application of the discharging method requires creative design of discharging rules.

An easy example

In 1904, Wernicke introduced the discharging method to prove the following theorem, which was part of an attempt to prove the four color theorem.

Theorem: If a planar graph has minimum degree 5, then it either has an edge with endpoints both of degree 5 or one with endpoints of degrees 5 and 6.

Proof: We use V, F, and E to denote the sets of vertices, faces, and edges, respectively. We call an edge *light* if its endpoints are both of degree 5 or are of degrees 5 and 6. Embed the graph in the plane. To prove the theorem, it is sufficient to only consider planar triangulations (for the following reason). We arbitrarily add edges to the graph until it is a triangulation. Since the original graph had minimum degree 5, each endpoint of a new edge has degree at least 6. So, none of the new edges are light. Thus, if the triangulation contains a light edge, then that edge must have been in the original graph.

We give the charge $6 - d(v)$ to each vertex v and the charge $6 - 2d(f)$ to each face f, where $d(x)$ denotes the degree of a vertex and the length of a face. (Since the graph is a triangulation, the charge on each face is 0.) Recall that the sum of all the degrees in the graph is equal to twice the number of edges; similarly, the sum of all the face lengths equals twice the number of edges. Using Euler's Formula, it's easy to see that the sum of all the charges is 12:

$$\sum_{f\in F} 6 - 2d(f) + \sum_{v\in V} 6 - d(v) =$$
$$6|F| - 2(2|E|) + 6|V| - 2|E| =$$
$$6(|F| - |E| + |V|) = \quad 12.$$

We use only a single discharging rule:

- Each degree 5 vertex gives a charge of 1/5 to each neighbor.

We consider which vertices could have positive final charge. The only vertices with positive initial charge are vertices of degree 5. Each degree 5 vertex gives a charge of 1/5 to each neighbor. So, each vertex is given a total charge of at most $d(v)/5$. The initial charge of each vertex v is $6 - d(v)$. So, the final charge of each vertex is at most $6 - 4d(v)/5$. Hence, a vertex can only have positive final charge if it has degree at most 7. Now we show that each vertex with positive final charge is adjacent to an endpoint of a light edge.

If a vertex v has degree 5 or 6 and has positive final charge, then v received charge from an adjacent degree 5 vertex u, so edge uv is light. If a vertex v has degree 7 and has positive final charge, then v received charge from at least 6 adjacent degree 5 vertices. Since the graph is a triangulation, the vertices adjacent to v must form a cycle, and since it has only degree 7, the degree 5 neighbors cannot be all separated by vertices of higher degree; at least two of the degree 5 neighbors of v must be adjacent to each other on this cycle. This yields the light edge.

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