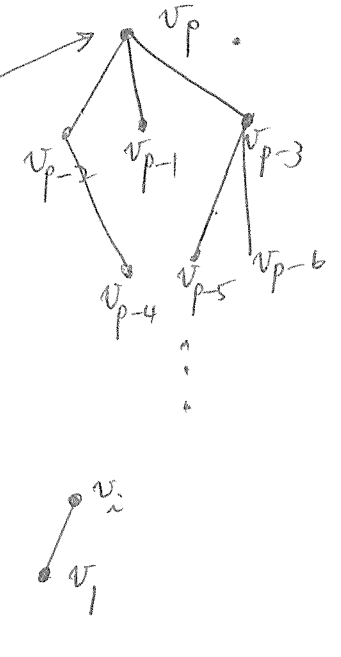
Brooks' Theorem

If G is a connected graph which is not a complete graph or an an odd cycle, then $\chi(G) \leq \Delta(G)$.

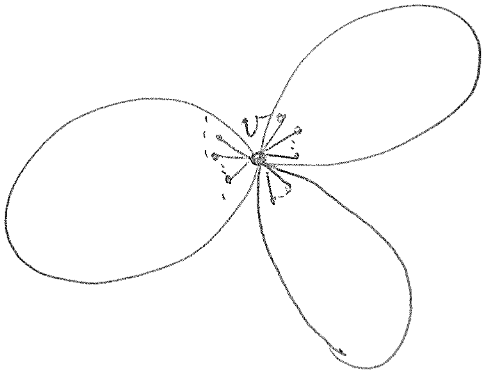
Proof. Clearly, it is true for $\Delta(G) \leq 2$. Let $\Delta(G) = k \geq 3$. First, if G is not k-regular, let $v \in V(G)$ such that $\deg(v) < k$. Let T be a spanning tree of G with root $v = v_p$ where $p = |G|$. Since every vertex v_i in $V(T) \setminus \{v_p\}$ has a unique $v_p - v_i$ path, the distance of v_p and v_i, $i = 1, 2, \cdots, p-1$ can be obtained. Following the length, we can arrange the vertices as $v_1, v_2, \cdots, v_p$ where $d(v_i, v_p) \geq d(v_{i+1}, v_p)$ for $i = 1, 2, \cdots, p-2$. Now, starting from v_1, at the time we color v_i, one of larger index v_j $(j > i)$ is not colored and thus at most $k-1$ vertices which have been colored. So, v_i can be colored. By greedy coloring, we end up with coloring v_p.



Since $\deg(v_p) < k$, at least one color is left to use.

On the other hand, let G be a k-regular graph.

Case 1. G has a cut-vertex v, i.e., $G-v$ is not connected.



Therefore, in each component G_i of $G-v$, $|\overbrace{N_G(v)\cap V(G_i)}^{N_i \overset{\text{def}}{=}}| < k$.

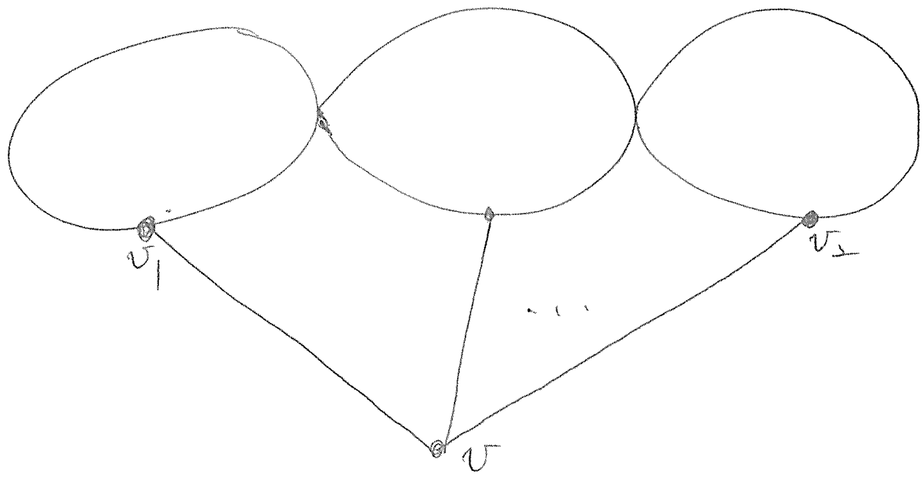
By the above argument, $\chi(G_i) \le k$. Now, by arranging the colors of N_i, we are able to find a color which did not occur in any N_i and we have a k-coloring of G.

Case 2. G is 3-connected.

Since G is not a complete graph, there exist three vertices v_1, v_2 and v such that $d(v_1,v_2)=2$, $d(v_1,v)=d(v_2,v)=1$, Moreover, $G-v_1-v_2$ is connected. In this case, we first color v_1 and v_2 by the same color, and then apply greedy coloring as above to obtain a k-coloring of G. (Notice here that we color v at last.)

Case 3. $\kappa(G-v)=1$

The graph can be depicted as follows.



We can choose v_1 and v_2 from different blocks and color them by the same color. The k-coloring of G can then be obtained by a similar way as in Case 2.

<u>Theorem</u> If G is an interval graph, then $\chi(G) = \omega(G)$.

<u>Proof</u>. Let $V(G) = \{v_1, v_2, \cdots, v_p\}$ such that $v_i = [l_i, r_i]$, moreover, let $l_1 \le l_2 \le \cdots \le l_p$. It suffices to prove that we need at most $\omega(G)$ colors to color the vertices of G. Starting from v_1, we apply <u>greedy coloring</u>.

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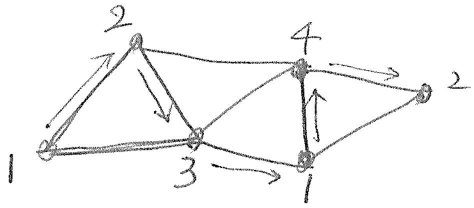
<u>Definition</u>

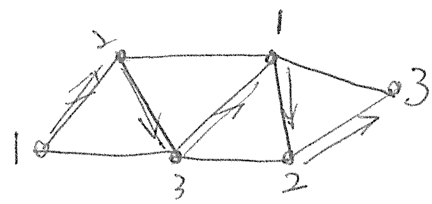
Let F be a collection of closed intervals on a real line. Then, the <u>intersection graph</u> of F is called an interval graph.

Greedy Coloring

Let the set of colors be $\{1, 2, \cdots, k\}$.
Start with color 1 and always choose the smallest integer for color we can use in each step, i.e., if we can use $1 \le i \le k$, then we don't use $j > i$.



No good!
(But it may happen!)



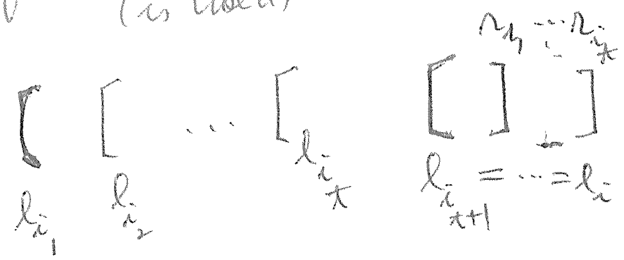
Good!
(Good luck!)

Greedy coloring
May not be able to obtain a $\chi(G)$-coloring for a graph G.

Clearly, if for a vertex v_i we need to use k, then there are $k-1$ vertices which are in $N(v_i) \cap \{v_1, v_2, \cdots, v_{i-1}\}$ and also the colors $1, 2, \cdots, k-1$ are used. Let them be $v_{i_1}, v_{i_2}, \cdots, v_{i_{k-1}}$. (We shall claim $\langle \{v_i, v_{i_1}, \cdots, v_{i_{k-1}}\} \rangle_G$ is a complete subgraph of order k.)

By assumption $l_{i_1} \le l_{i_2} \le \cdots \le l_{i_{k-1}} \le l_i$. Now, if these k real numbers are equal, we have the claim. Otherwise, let $t \in \{1, 2, \cdots, k-1\}$ such that $l_{i_t} < l_i$, but $l_{i_{t+1}} = l_i$, i.e. $l_{i_{t+1}} = l_{i_{t+2}} = \cdots = l_{i_{k-1}} = l_i$. Now, we a clique obtained from $\{v_{i_{t+1}}, \cdots, v_{i_{k-1}}, v_i\}$. But, now, for $1 \le j \le t$, $v_{i_j} \sim v_i$. So, $r_{i_j} > l_i$ for each $1 \le j \le t$. This implies that $l_i \in [l_{i_h}, r_{i_h}]$ for $h = 1, 2, \cdots, k-1$ and thus we have the $\overset{\text{(claim)}}{\text{proof}}$.

Since if k occurs (is used), then we have a complete subgraph of order k. Hence $\chi(G) \le \omega(G)$.



The other inequality is easy to see.

<u>Definition</u> (Chordal graph or Triangulated graph)
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A graph is called a chordal graph if every cycle of length larger than 3 has a chord.

<u>Theorem</u>

If G is a chordal graph, then $\chi(G) = \omega(G)$.

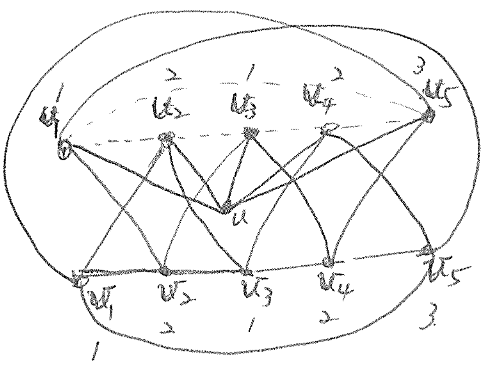
<u>Proof.</u> (Exercise)

(*) A graph without a C_3 can be of very large chromatic number.

<u>Theorem</u> (Mycielski, 1955)

For every positive integer n, there exists a graph G which is triangle-free and $\chi(G) = n$.

<u>Proof.</u> $n = 1, 2, 3$, G can be chosen as K_1, K_2 and C_5 resp.

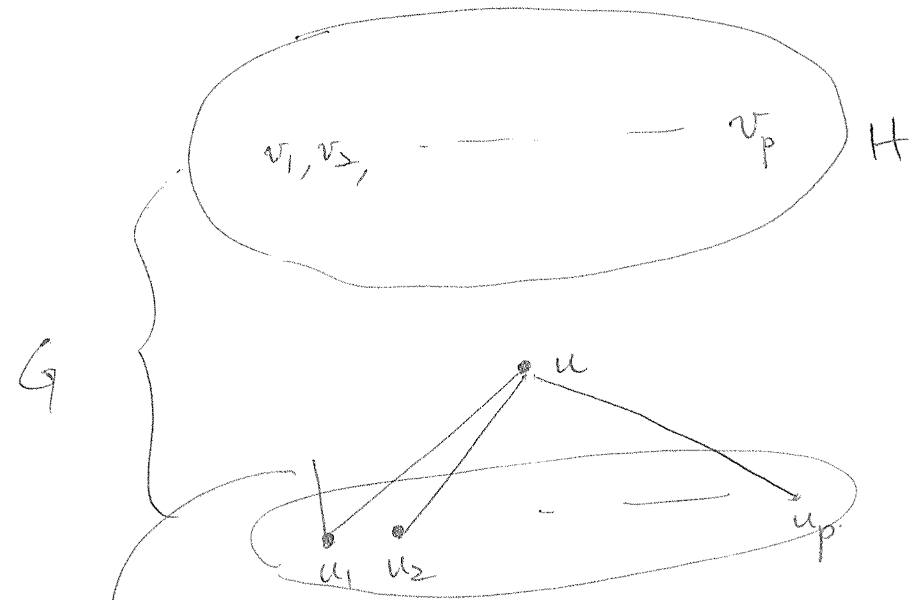


$\leftarrow$ u_i Assign the same color as v_i

$\varphi(u) = 4$

This graph is 4-colorable but not 3-colorable.

If G has a 3-coloring and $\varphi(u) = 3$, then some v_i's are colored 3, say v_5. Now, recolor v_5 by using the color of u_5, we have a 2-coloring of C_5, a contradiction.



incident to each vertex of $N_H(v_1)$

u_i ———————— $N_H(v_i)$

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