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Theorem (Fu et. al, JGT, 1997)

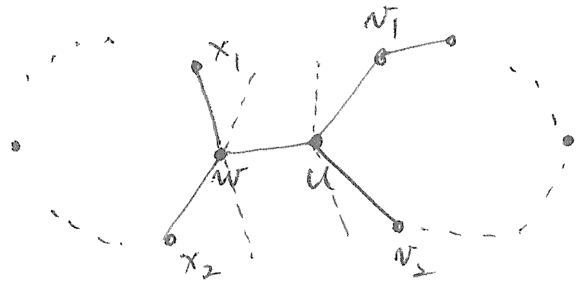
For all $k \geq 3$ and $3 \leq g_1 < g_2$, $f(k; g_1) < f(k; g_2)$.

Proof.

It suffices to show that if $k, g \geq 3$ then $f(k;g) < f(k;g+1)$.
Let G be a $(k; g+1)$-graph with $f(k; g+1)$ vertices.

Case 1. k is odd.

Let C be a cycle of length $g+1$ in G containing uv_1 and uv_2. Let $N_G(u) = \{v_1, v_2, \cdots, w\}$. w exists since $k \geq 3$. Now, let $w \notin N_G(u)$. For otherwise, we have a shorter cycle. $N_G(w) = \{x_1, x_2, \cdots, x_{k-1}, u\}$ and G' be a component of $G - \{u, w\} + \{v_{2i-1}v_{2i},\ x_{2i-1}x_{2i} \mid 1 \leq i \leq \frac{k-1}{2}\}$ that contains v_1.

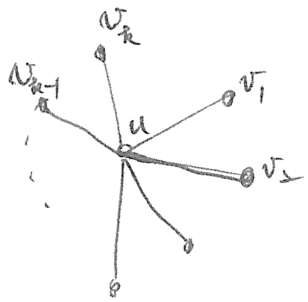


Since $g+1 \geq 4$, $N_G(u)$ and $N_G(w)$ are independent sets of G, so G' is a simple graph. Moreover, $C - u + v_1v_2$ is a cycle of length g and no cycles of G' has length less than g.
Hence G' is a $(k; g)$-graph with $f(k; g+1) - 2$ vertices and we have

$$f(k;g) \leq |G'| = f(k;g+1) - 2 < f(k;g+1).$$

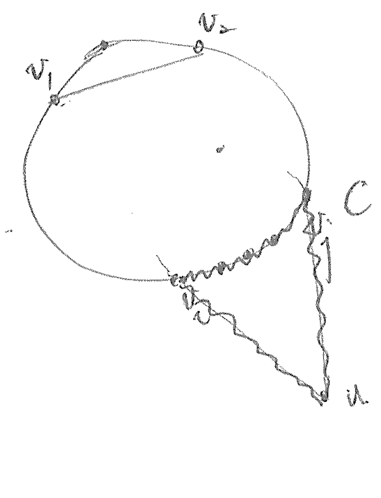
Case 2 k is even.

Let C be a cycle of length $g+1$ containing uv_1 and uv_2 (similar to case 1) and $N_G(u) = \{v_1, v_2, \cdots, v_k\}$.



Since $N_G(u)$ is an independent set, $G - u + \underbrace{\{v_1v_2, v_3v_4, \cdots, v_{k-1}v_k\}}_{E'}$ is a simple graph. Again, let G' be a component which contains $C - u + v_1v_2$, a cycle of length g.

Notice that if C' is a cycle in G' such that $E(C') \cap E' = \phi$, then C' is a cycle in G, hence $|E(C')| \geq g+1$. On the other hand, if $E(C') \cap E' \neq \emptyset$ then let P be a (v_i, v_j)-path that is a subgraph of C' with $E(P) \cap E' = \phi$. Now, $P + \{uv_i, uv_j\}$ is a cycle in G. This implies that $|C'| \geq g$. So G' is a $(k; g)$-graph with $f(k; g+1) - 1$ vertices.



Hence

$$f(k; g) < f(k; g+1).$$

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