The idea of maximal counterexample

(10, 2)

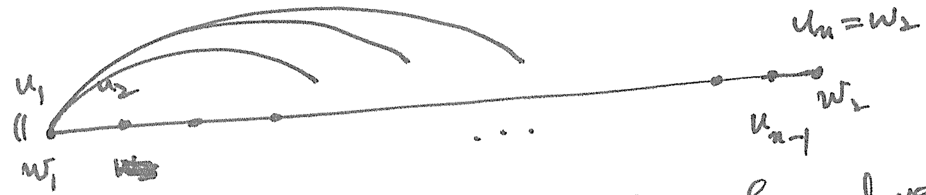
Theorem (Ore, 1960 AMM)

If a graph of order $n \geq 3$ such that for all distinct nonadjacent vertices u and v, $\deg(u) + \deg(v) \geq n$, then G is hamiltonian.

Proof: Assume the theorem is false. Hence there exists a "maximal nonhamiltonian" graph G of order $n \geq 3$ that satisfies the assumption of the theorem. This implies that ——— (1)

$\forall\ w_1 \not\sim_G w_2$, $G + w_1w_2$ is hamiltonian.

(Note that such pair of vertices w_1 and w_2 does exist since G is not hamiltonian.)



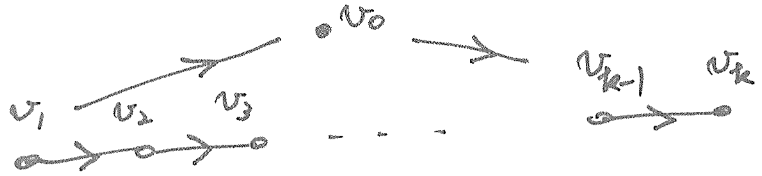
By (1), we have a hamiltonian path with end vertices w_1 and w_2, see above figure. Now, consider the neighbors of w_1 and w_2 respectively. Let $N(w_1) = \{u_{i_1}, u_{i_2}, \cdots, u_{i_k}\}$ with $\underset{\underset{w_2}{\|}}{i_1} < i_2 < \cdots < i_k \leq n-1$. Clearly, $\{u_{i_1-1}, u_{i_2-1}, \cdots, u_{i_k-1}\}$ $\cap\, N(w_2)$ is an empty set, so, $\underset{\underset{\deg(w_1)}{\|}}{|N(w_1)|} + \underset{\deg(w_2)}{|N(w_2)|} \leq n-1$. (?) $\rightarrow\leftarrow$.

Theorem (Tournament's fact) (Rédei, 1934)

Every tournament T contains a hamiltonian path. (digraph)

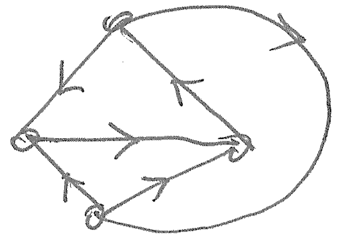
Proof: direction

Let $\langle\!\langle v_1, v_2, \cdots, v_k \rangle\!\rangle$ be the longest path in a tournament T. See the following figure for reference.



If $k = |T|$, then we are done.
Otherwise, assume that $k < |T|$. Then, $\exists\ v_0 \in V(T)$ which is not on $\langle\!\langle v_1, \cdots, v_k \rangle\!\rangle$. By assumption, (v_1, v_0) and (v_0, v_k) are arcs in T. This also implies that $\exists\ 2 \le i \le k-1$, such that (v_i, v_0) and (v_0, v_{i+1}) are arcs in T. In this case, we have a longer path, namely, $\langle\!\langle v_1, v_2, \cdots, v_i, v_0, v_{i+1}, \cdots, v_k \rangle\!\rangle$. $\rightarrow\leftarrow$.

Say some more about tournaments! Can you?



Connectivity

Definition (Vertex-connectivity)

The vertex-connectivity or simply connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal from G results in a disconnected or trivial graph.

Definition (Edge-connectivity)

The edge-connectivity $\kappa_1(G)$ of a graph G is the minimum number of edges whose removal from G results in a disconnected (or trivial graph).

Theorem $\forall$ graph G, $\kappa(G) \leq \kappa_1(G) \leq \delta(G)$.

Proof. (Easy to see?)

Question $\forall\ 1 \leq a \leq b \leq c$, construct a graph G such that $\kappa(G) = a$, $\kappa_1(G) = b$ and $\delta(G) = c$.