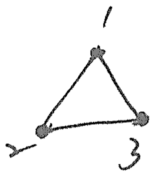
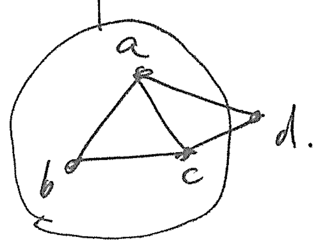
9.18

General sense of subgraphs

A graph G' is a $\boxed{\text{subgraph}}$ of G if G' is isomorphic to to a subgraph of G.



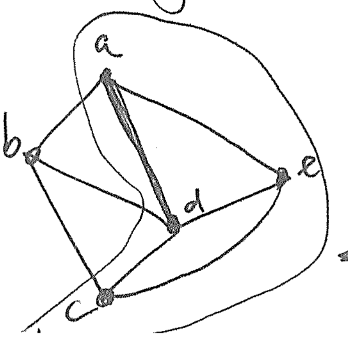
G:


(*) We shall say: G contains a triangle (or $\underset{\sim}{H}$) as the case may be.

say $\boxed{C_4}$

Induced subgraph

Let S be a subset of $V(G)$. Then the subgraph obtained from the vertex set S and all the edges in G ~~⟨S⟩G~~ which are connecting vertices in S is called the ~~an~~ induced subgraph of G by S, denoted by $\langle S\rangle_G$.

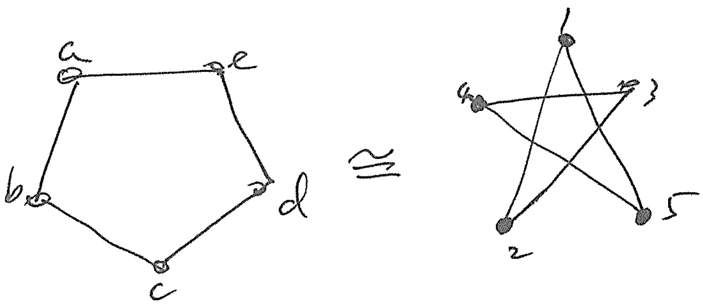
G:


$\langle\{a,d,c,e\}\rangle_G$

Subgraph

$G' = (V', E')$ is a subgraph of $G = (V, E)$ provided that $V' \subseteq V$ and $E' \subseteq E$.

Two graphs G and H are isomorphic if there exists a bijection $\varphi: V(G) \longrightarrow V(H)$ such that $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(H)$.



An automorphism of a graph G is an isomorphism from G onto itself.

e.g. $\varphi(i) = i+1 \pmod 5$

is an isomorphism from C_5 onto C_5

which is an automorphism of C_5.

The set of all automorphisms of G is denoted by $Aut(G)$ and an $Aut(G)$ exists for all graphs G. In fact, $\langle Aut(G), \circ \rangle$ is a group, called automorphism group.

Graph Structure problems

Given an "H", for which graphs G of order n, G must contain a subgraph H.

e.g. $H = C_3$.

Can you prove that if G has more than $\frac{n^2}{4}$ edges then G has a C_3?

<u>Definition</u> A graph G is said to be H-free if G does not contain H as a subgraph.

<u>Research problem</u>

Find a graph G of order n which has the maximum number of edges, but G is H-free.

e.g. If $G = C_3$, then $ext(n; C_3) = \lfloor \frac{n^2}{4} \rfloor$.

9,23 1

<u>Notations</u>

G

$\Delta(G)$: maximum degree

$\delta(G)$: minimum degree

$d(G)$: average degree

$$\delta(G) \le d(G) \le \Delta(G)$$

$$\varepsilon(G) \; \boxed{: \tfrac{1}{2} d(G)}$$

$\|_{def}$

$\|G\| / |G|$

proof.

$$d(G) = \frac{1}{|G|} \sum_{v \in V(G)} \deg_G(v)$$

$$\|G\| = \frac{1}{2} \sum_{v \in V(G)} \deg_G(v) = |G| \cdot \frac{1}{2} d(G)$$

$$\frac{\|G\|}{|G|} = \frac{1}{2} d(G) = \varepsilon(G) \quad \text{(average size on one vertex)}$$

<u>Subgraph</u>

P_n : a path of order n and length $n-1$
C_n : a cycle of order n

<u>proposition</u>

Every graph contains a path of length $\delta(G)$ and a cycle of length $\delta(G)+1$ (provided $\delta(G) \ge 2$).

<u>Proof.</u> Let $\langle x_0, x_1, \cdots, x_k \rangle$ ba longest path we can find in G. Then all neighbors of x_k lie on the path $\langle x_0, x_1, \cdots, x_{k-1} \rangle$. Hence $k \ge d(x_k) \ge \delta(G)$. Now, if $\delta(G) \ge 2$, then x_k is adjacent to at least one vertex on $\langle x_0, \cdots, x_{k-1} \rangle$. Let i be the minimal with $x_k x_i \in E($

Then $(x_i, x_{i+1}, \cdots, x_k)$ is a cycle. Since $d(x_k) \geq \delta$, $i \leq k-\delta$.
This implies the the cycle has at least $\delta+1$ vertices. ▨

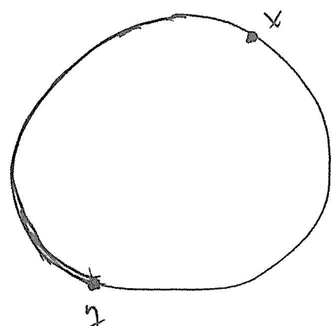
[illegible] $\boxed{d_G(x,y)}$: shortest path, $= \infty$, if no paths join them.

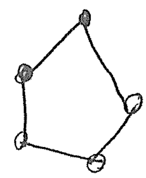
Proposition

Every graph G containing a cycle satisfies $g(G) \leq 2\,\mathrm{diam}(G)+1$.

Proof.

Let C be a shortest cycle in G. If $g(G) \geq 2\,\mathrm{diam}(G)+2$, then C has two vertices x and y whose distance in "C" is at least $\boxed{d_C(x,y)}$ $\mathrm{diam}(G)+1$. But, in G, these two vertices have a lesser distance; any shortest path P between x and y is not a subgraph of C.





$e_G(x)$: eccentricity of x (離心率).

$\Vert$ $\max_{y \in V(G)} d(x,y)$

$\mathrm{diam}(G) = \max_{x \in V(G)} e_G(x)$, $\mathrm{rad}(G) = \min_{x \in V(G)} e_G(x)$

$\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\,\mathrm{rad}(G)$

☆ A vertex x is <u>central</u> if $e_G(x) = rad(G)$.

<u>Proof.</u>

$$diam(G) \leq 2\, rad(G)$$

Let x, y be two vertices in G such that $d_G(x,y) = diam(G)$. Let w be a central vertex, i.e. $e_G(w) = rad(G)$. Since $d_G(x,y) \leq d_G(x,w) + d_G(w,y)$, (<u>$d$ is a metric</u>), $d_G(x,y) \leq e_G(w) + e_G(w) = 2\, rad(G)$.

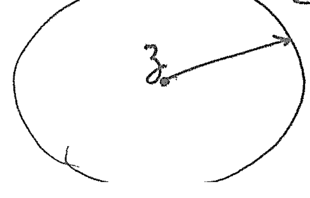
<u>Problem</u>

Let a, b be two positive integers such that $a \leq b \leq 2a$. Then, find a graph G ~~whose~~ with $rad(G) = a$ and $diam(G) = b$.

e.g. $a = 30$, $b = 50$.

<u>Thm.</u> A graph G ~~of radius at most k and maximum~~ with $rad(G) \leq k$ and $\Delta(G) \leq d (\geq 3)$ has fewer than $\frac{d}{d-2}(d-1)^k$ vertices, i.e. $|G| \leq \frac{d}{d-2}(d-1)^k$.

<u>Proof.</u> Let z be a central vertex.



$D_i = \{y \mid d_G(y,z) = i\}$

$|D_0| = 1$, $|D_1| \leq d$, $|D_{i+1}| \leq (d-1)|D_i|$

$|D_{i+1}| \leq d(d-1)^i$ for $i < k$.

$$|G| \le 1 + D_1 + D_2 + \cdots + D_k$$

$$= 1 + \sum_{i=0}^{k-1} d(d-1)^i$$

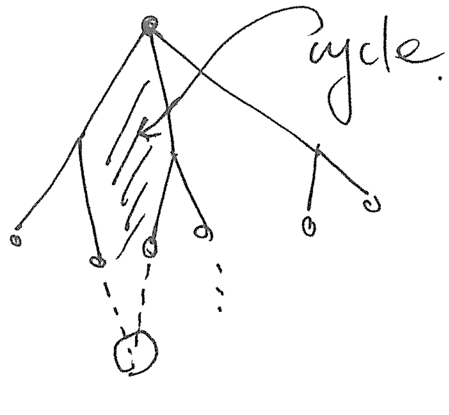
$$= 1 + d \sum_{i=1}^{k-1} (d-1)^i$$

$$= 1 + \frac{d}{d-2}\left((d-1)^k - 1\right) < \frac{d}{d-2}(d-1)^k. \quad \blacksquare$$

(*) ~~If~~ we use min. degree $\delta(G)$ and $g(G)$, then we can find the lower bound of $|G|$.

<u>Theorem</u> (Alon, Hoory and Linial, 2002) Let $d(G) \ge d \ge 2$ and $g(G) \ge g \in \mathbb{N}$, then

$$|G| \ge n_0(d,g) := \begin{cases} 1 + d\sum_{i=0}^{r-1}(d-1)^i, & \text{if } g = 2r+1; \\ 2\sum_{i=0}^{r-1}(d-1)^i, & \text{if } g = 2r. \end{cases}$$



Now, consider 3-regular graph with girth g.

$g=5$. $n_0(3,5) = 1+3\cdot\sum_{i=0}^{1}(3-1)^i$

$= 1+3\cdot(1+2) = 10$

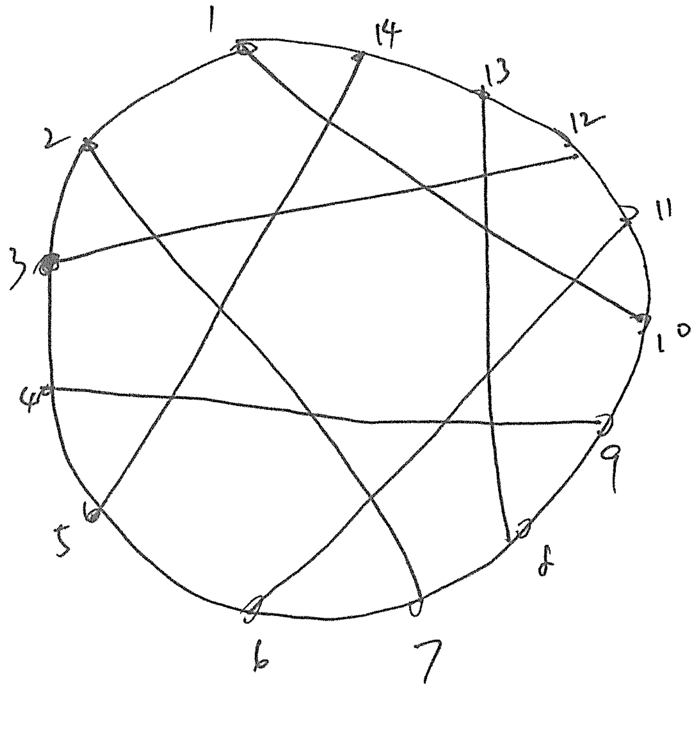
$|G| \geq \underline{10}$.

5-cage

$g=6$, $n_0(3,6) = 2\sum_{i=0}^{2}(3-1)^i$

$= 2\cdot(1+2+4) = \underline{\underline{14}}$

$|G| \geq 14$

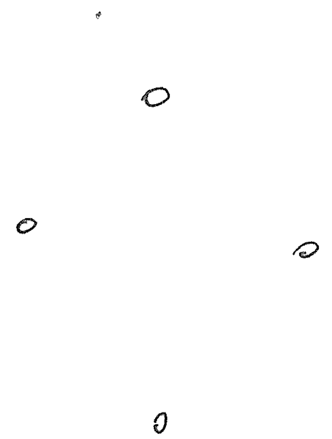


6-cage

$d = 4 \quad g = 5$

$$|G| \geq 1 + 4 \sum_{i=0}^{1} (4-1)^i$$

$$= 1 + 4(1+3) = 17$$



The existence of cycles

9.25 1

Fact: If G contains a cycle, and G is connected, then $\|G\| \geq |G|$.

If $\delta(G) \geq 2$, then G contains a cycle.

(Weaker) If $d(G) \geq 2$, then G contains a cycle.

$\delta(G) \geq 2 \Rightarrow d(G) \geq 2$.

⑥下周讲 下页

Question When does a graph can be decomposed into cycles?

Definition A graph G can be decomposed into subgraphs in $\mathcal{H} = \{H_1, H_2, \cdots, H_k\}$ if $E(G)$ can be partitioned into sets $E_1, E_2, \cdots, E_\ell$ such that for each $i = 1, 2, \cdots, n$ $\langle E_i \rangle_G \cong H_j$ for some $j \in \{1, 2, \cdots, k\}$. If $\mathcal{H}$ contains a single graph H, then we write $H | G$ and say G has an H-decomposition. If G is K_n, then an H-decomposition of K_n is also known as an H-design of order n.

e.g. A C_3-design of order 7 exists.

A C_4-design of order 9 exists.

Come back to the question. 9,25 2

Theorem 1 If G is an even graph, then G can be decomposed into cycles.

---

t-Euler

Eulerian circuit theorem $\boxed{(1736)}$ 最早的圖论论文

① G has an eulerian circuit if and only if G is connected and even.

② A digraph G has an eulerian circuit if and only if G is connected and balanced ($\forall$ each vertex v,

$\underset{\text{outdegree}}{deg_G^+(v)} = \underset{\text{indegree}}{deg_G^-(v)}$.)

Theorem 1 is a corollary of Eulerian Circuit Theorem.

This theorem is good for multi-graphs (graphs with multiple edges and  is considered as a 2-cycle.

Proof. By induction on $\|G\|$.

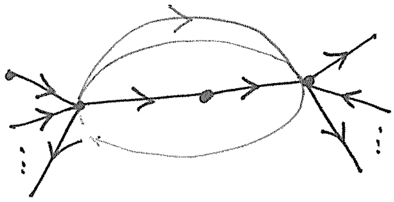
Case 1. $\delta(G) = 2$. (Omit the vertex.)

Case 2. $\delta(G) \geq 4$. (Delete a cycle.)

为了证明容易写。 $\blacksquare$

Directed case

Case 1. $\exists v,\ \deg^+_G(v) = 1 = \deg^-_G(v)$: (Omit the vertex).



Case 2. $\delta^+(G) = \delta^-(G) \geq 2$.
(Delete a directed cycle.)

Question 1 How to find the circuit ?
(It is easy to prove, but not easy to find a "realistic" circuit.)

Question 2 How many different (distinct) circuits are there in G ?

(1) Directed case: Solved

(2) undirected case: Unsolved !

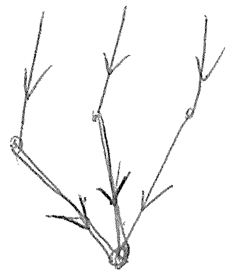
Best Theorem

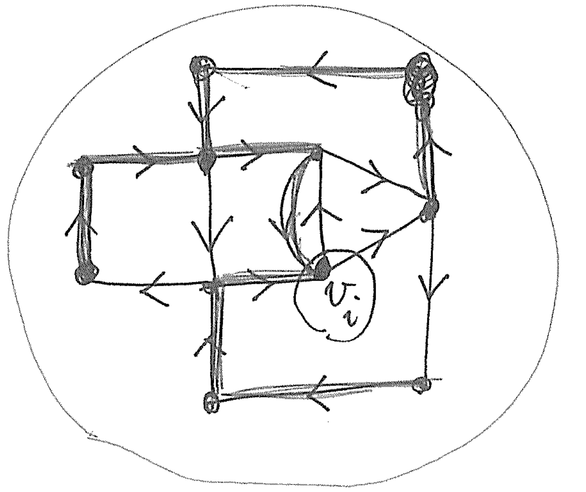
(de Bruijn, van Aardenne-Ehrenfest, Smith and Tutte)

Let G be a balaned "connected" digraph with $V(G) = \{v_1, v_2, \dots, v_n\}$. Then the number of eulerian circuits of G, denoted by $s(G)$, is equal to

$$t_i(G) \prod_{j=1}^{n} (d^+(v_j)-1)! \text{ where } t_i(G) \text{ is the number}$$

of spanning trees oriented to v_i.
toward





In fact, $t_1(G) = t_2(G) = \cdots = t_n(G)$.

Infinite case ($\|G\| = +\infty$).

Theorem

Let $G=(V,E)$ be a connected infinite multigraph with E infinite. Then G has an infinite eulerian circuit if and only if

(1) E is countable,

(2) every vertex is of even degree or infinite, and

(3) for ~~every~~ subgraph $G' \subseteq G$, $G' = (V', E')$ with E' finite, the graph $G - E'$ has at most two infinite components; furthermore, if $d_{G'}(x)$ is even for every vertex $x \in V'$, then $G - E'$ has precisely one infinite component.

Fleury's Algorithm Given an eulerian graph G:

Step 1. Let $i=0$ and select an arbitrary vertex v_0 of G and define $T_0 : v_0$.

Step 2. Given that the trail $T_i : v_0, e_1, v_1, \ldots, e_i, v_i$, has been constructed, select an edge e_{i+1} from $E(G) - \{e_1, \ldots, e_i\}$ subject to the conditions: (a) e_{i+1} is incident ~~to~~ with v_i; (b) unless there is no other choice, e_{i+1} is not a bridge of the graph $G_i = G - \{e_1, \ldots, e_i\}$.
If no such e_{i+1} exists, stop.

Step 3. Define $T_{i+1} : T_i, e_{i+1}, v_{i+1}$. Step 4. Replace i by $i+1$ and return to Step 2.