Ascending Subgraph Decomposition of Graphs

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Graph Decomposition

- The topic of this talk is about graph decomposition.
- A graph G is said to be decomposed into subgraphs G_1, G_2, …, G_k if the edge set of G can be partitioned into k subsets E_1, E_2, …, E_k such that these k sets induce the above k subgraphs respectively.
- For convenience we use $G = G_1 + G_2 + \ldots + G_k$ to denote the decomposition.

Ascending Subgraph Decomposition

- If $G = G_1 + G_2 + \ldots + G_k$ and G_i is a proper subgraph of G_{i+1} for $i = 1, 2, \ldots, k-1$, then G is said to have an ascending subgraph decomposition with k subgraphs.
- ***Conjecture*** *(posed by Alvai et al. 1987)*

If G is a graph of size $n(n+1)/2 \leq |E(G)| < (n+1)(n+2)/2$, then G has an ascending subgraph decomposition with n subgraphs.

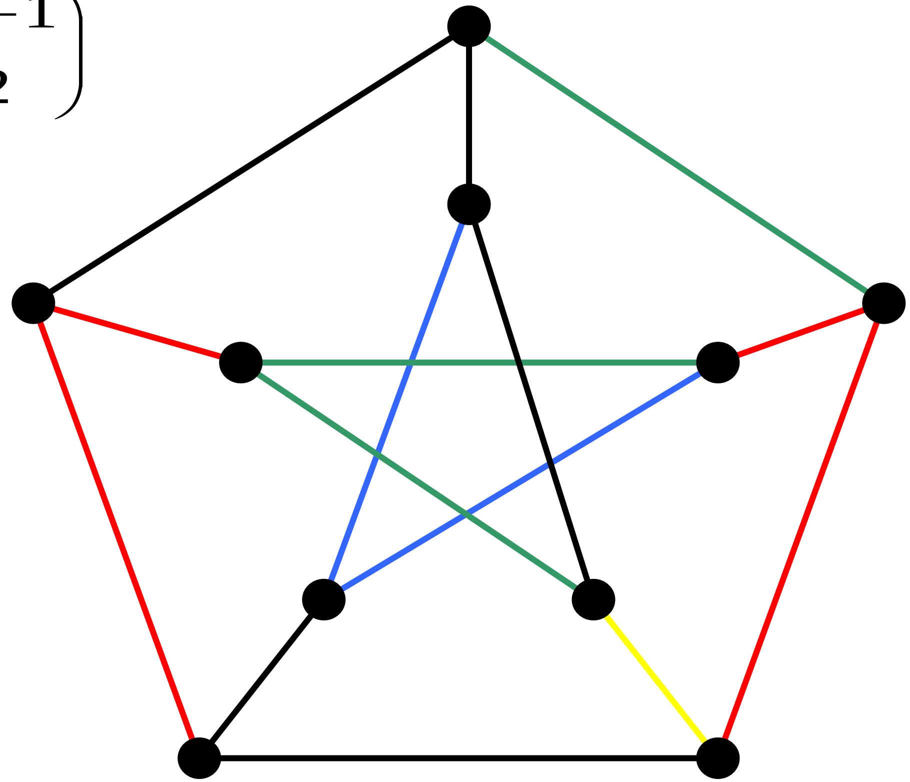
Problem with Value

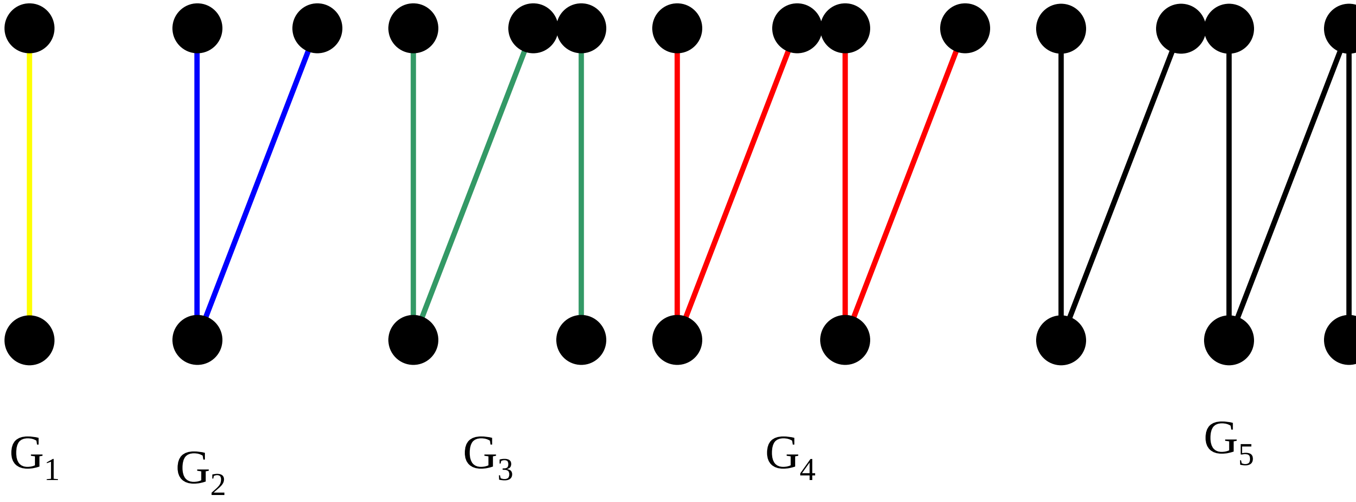
- This problem is known as the ASD problem.
- This problem is also one of the priced problems introduced to the researchers of graph theory by P. Erdös in his famous talks "Unsolved problems".

Another version of ASD Problem

- ***Conjecture*** *If G is a graph of size n(n+1)/2, then G has an ascending subgraph decomposition with n subgraphs.*
- In the case, the subgraph G_i has exactly i edges *for i = 1, 2, …, n.*
- If we can solve this version, then the original version is also solved, see it?

Example: $15=\binom{5+1}{2}$

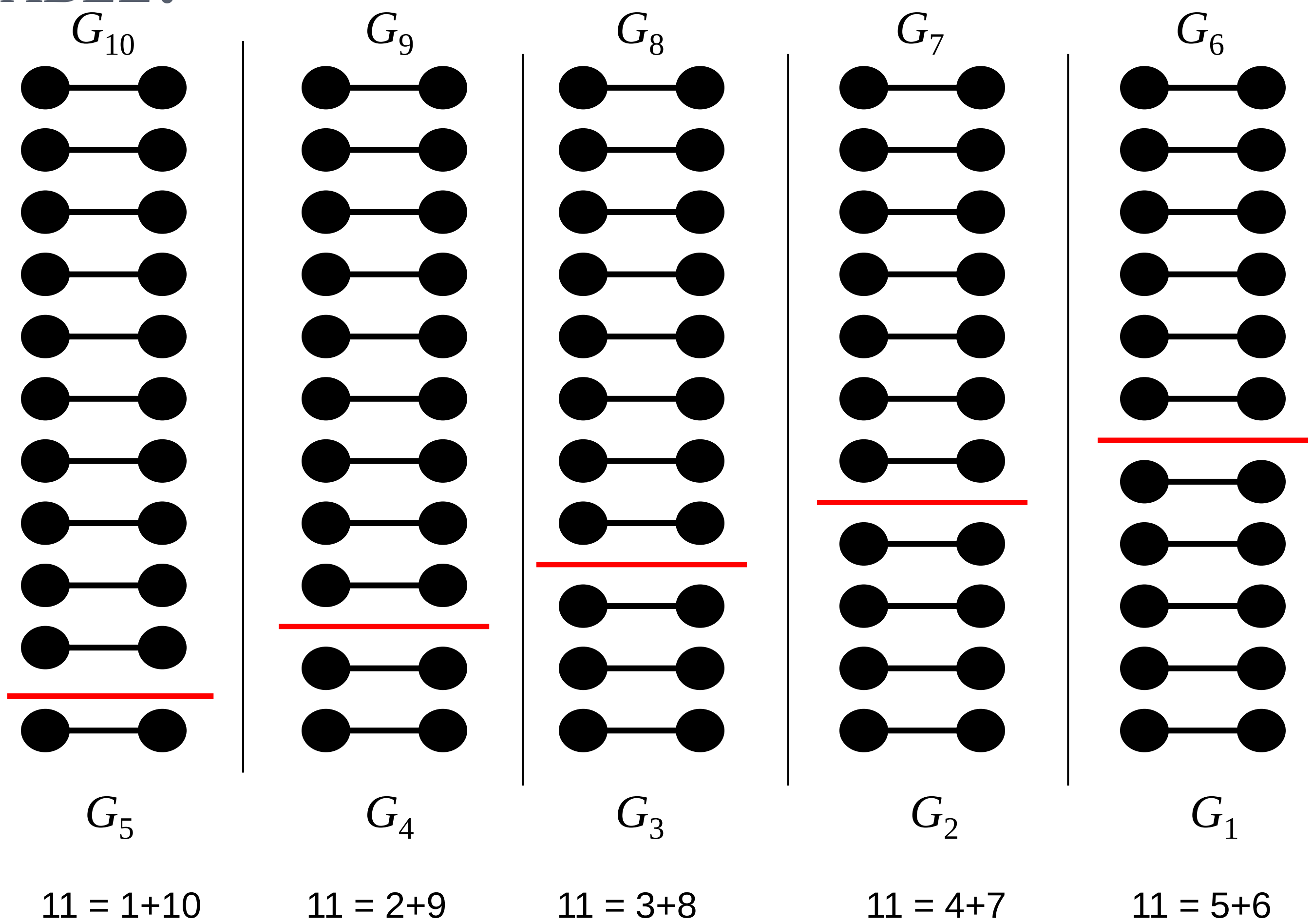




More known results

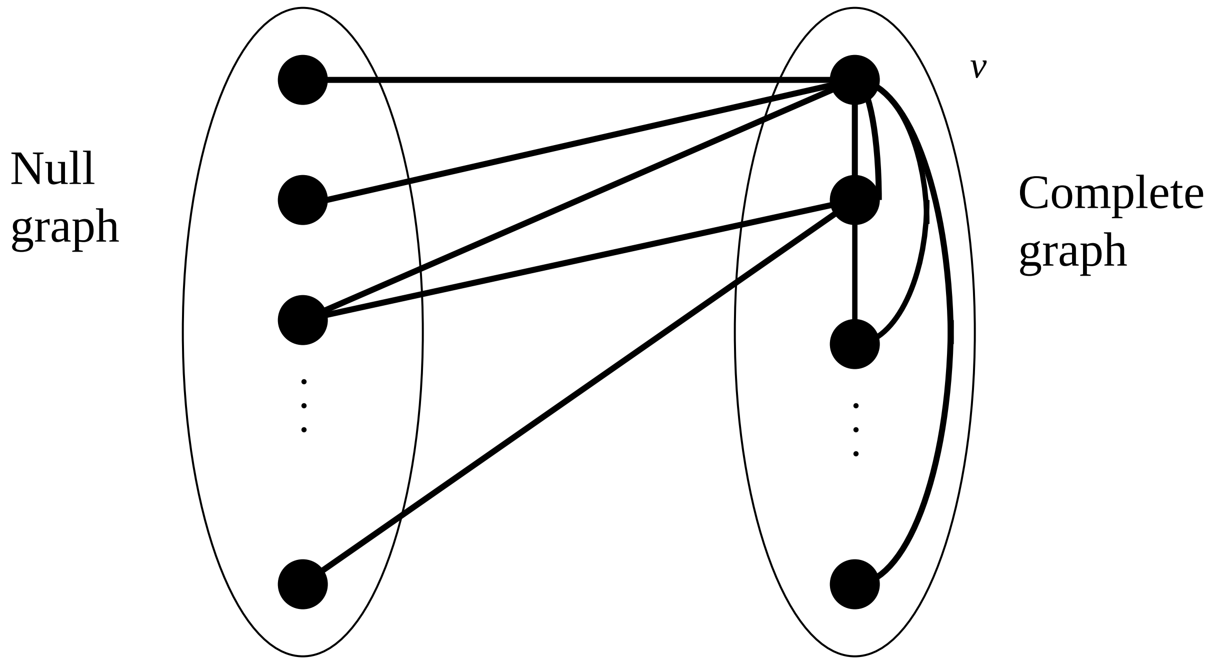
- Star forests (Two papers)
- $\Delta(G) \leq (n-1)/2$ (Each member is a matching)
- Split graphs
- $\Delta(G) \leq (2 - 2^{1/2})n$
- Nearly complete graphs
- Regular graphs
- Complete multipartite graphs

Example : $|E(G)| = \binom{10+1}{2}$ and G is 5 edge-colorable.



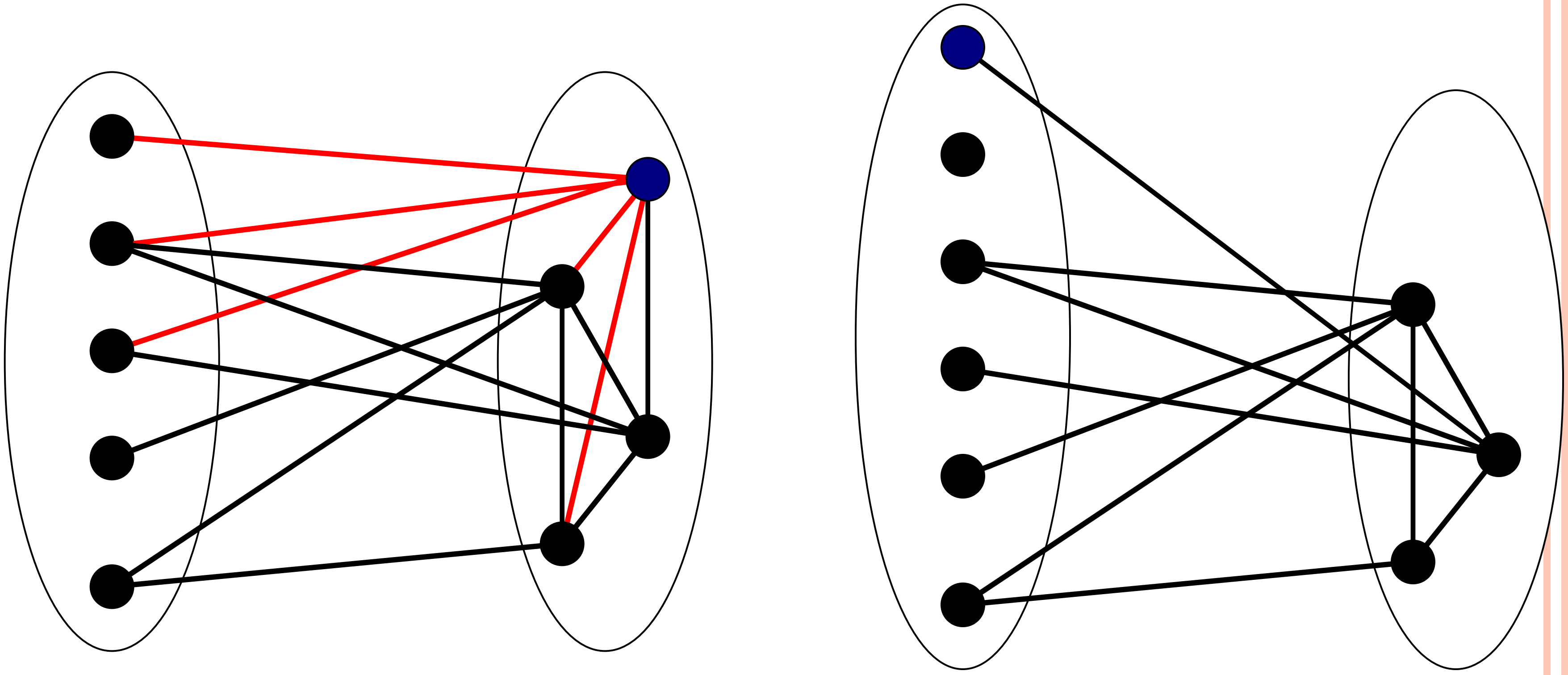
G_i = a matching of size i for I = 1, 2, …, 10

ANY SPLIT GRAPH ON $\binom{n+1}{2}$ EDGES HAS AN *ASD*.



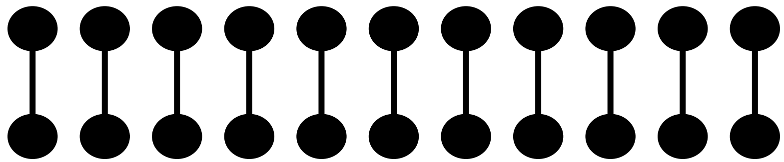
Proof : Delete a star of *n* edges from the edges from the edges incident to *v* (the edges between null graph and complete graph first) and the by induction.

Example : $|E(G)| = \binom{5+1}{2}$

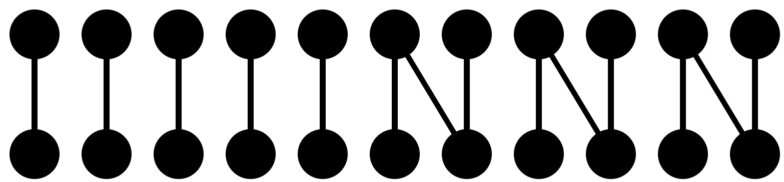


Any r-regular graph G on $\binom{n+1}{2}+t$ edges where $t < n$, has an *ASD*.

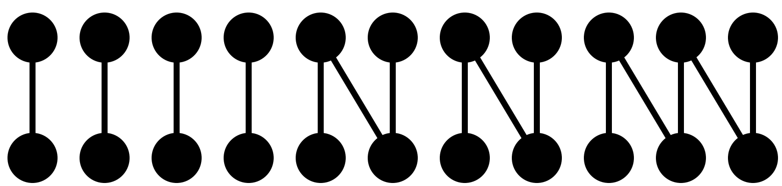
Case 1. $r \leq n/2$



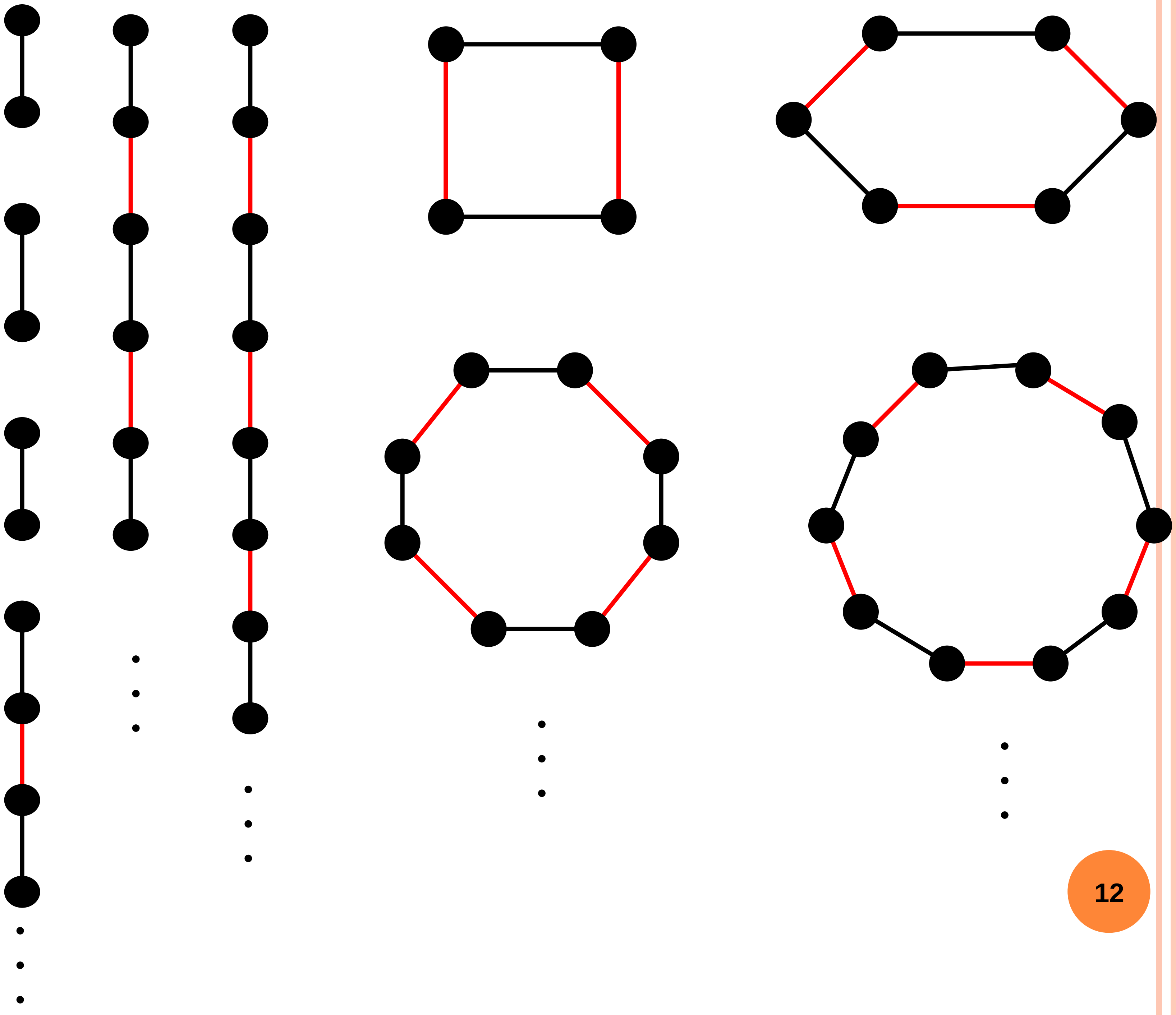
Case 2. $n/2<r \leq 2n/3$:



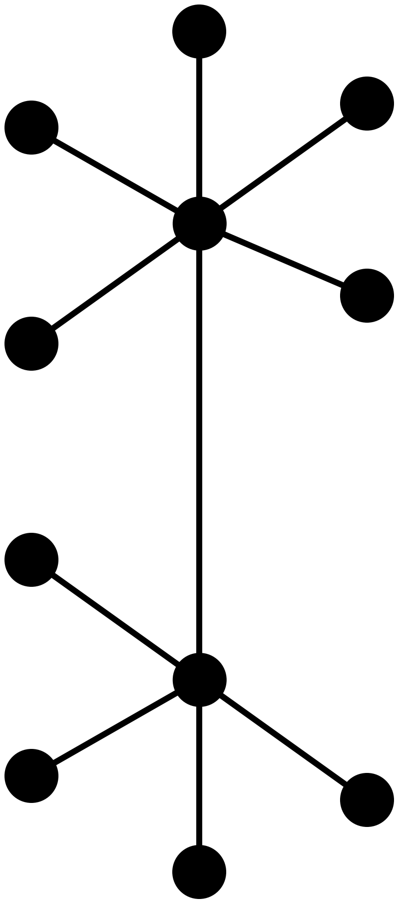
Case 3. $2n/3<r<v/2$:



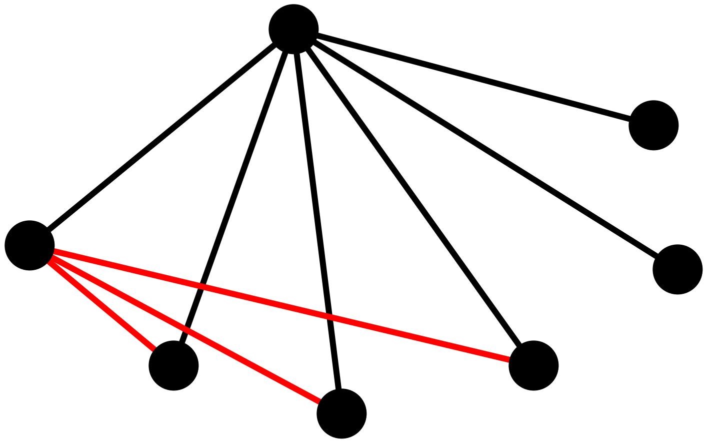
Case 4. $r \geq v/2$. Peel off Hamiltonian cycles from the graph until the remaining valency $r' < v/2$ and the members G_i would be linear forest.



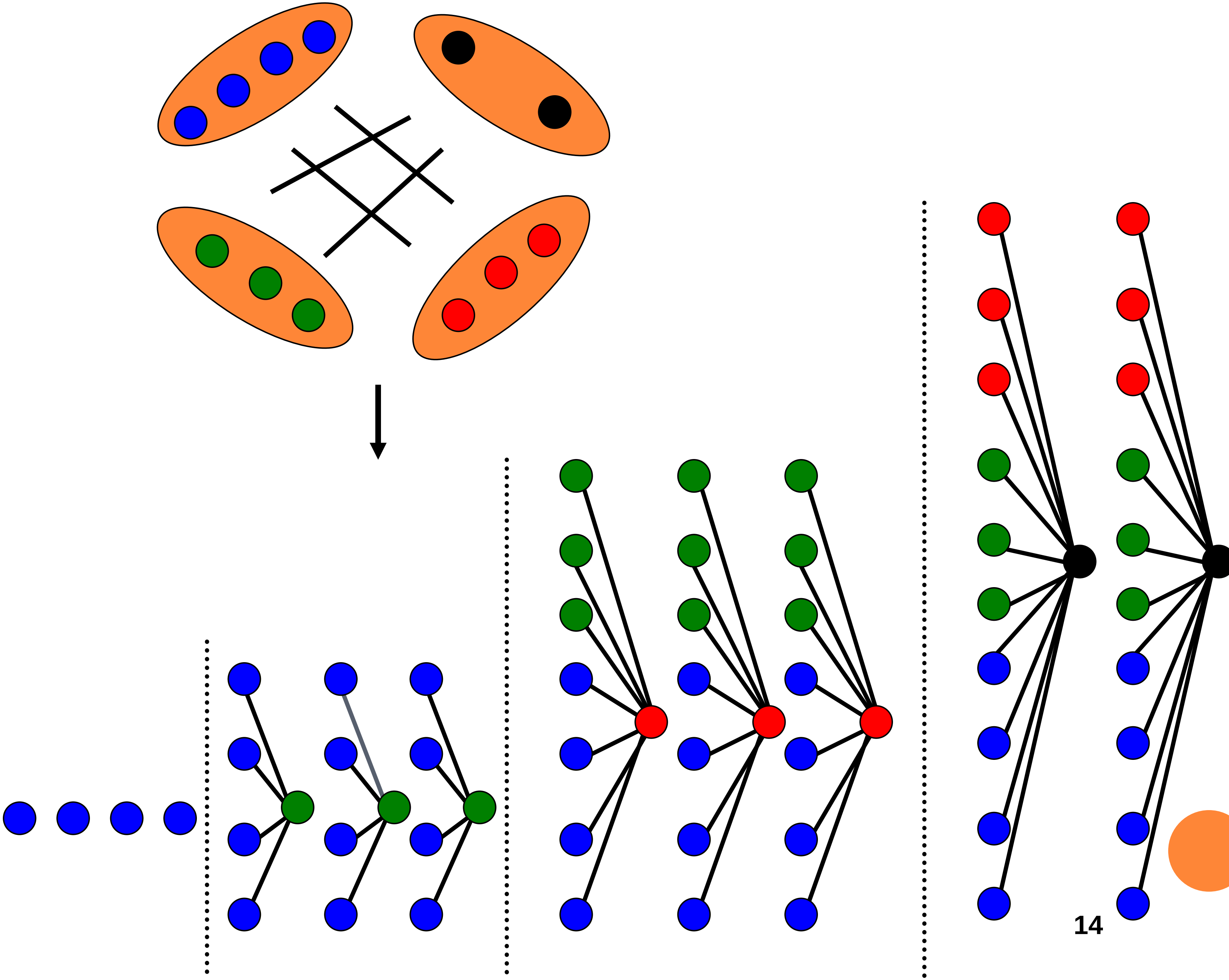
Any complete multipartite graph has an ASD with each member a star or a double star or a pregnant star.



Double star



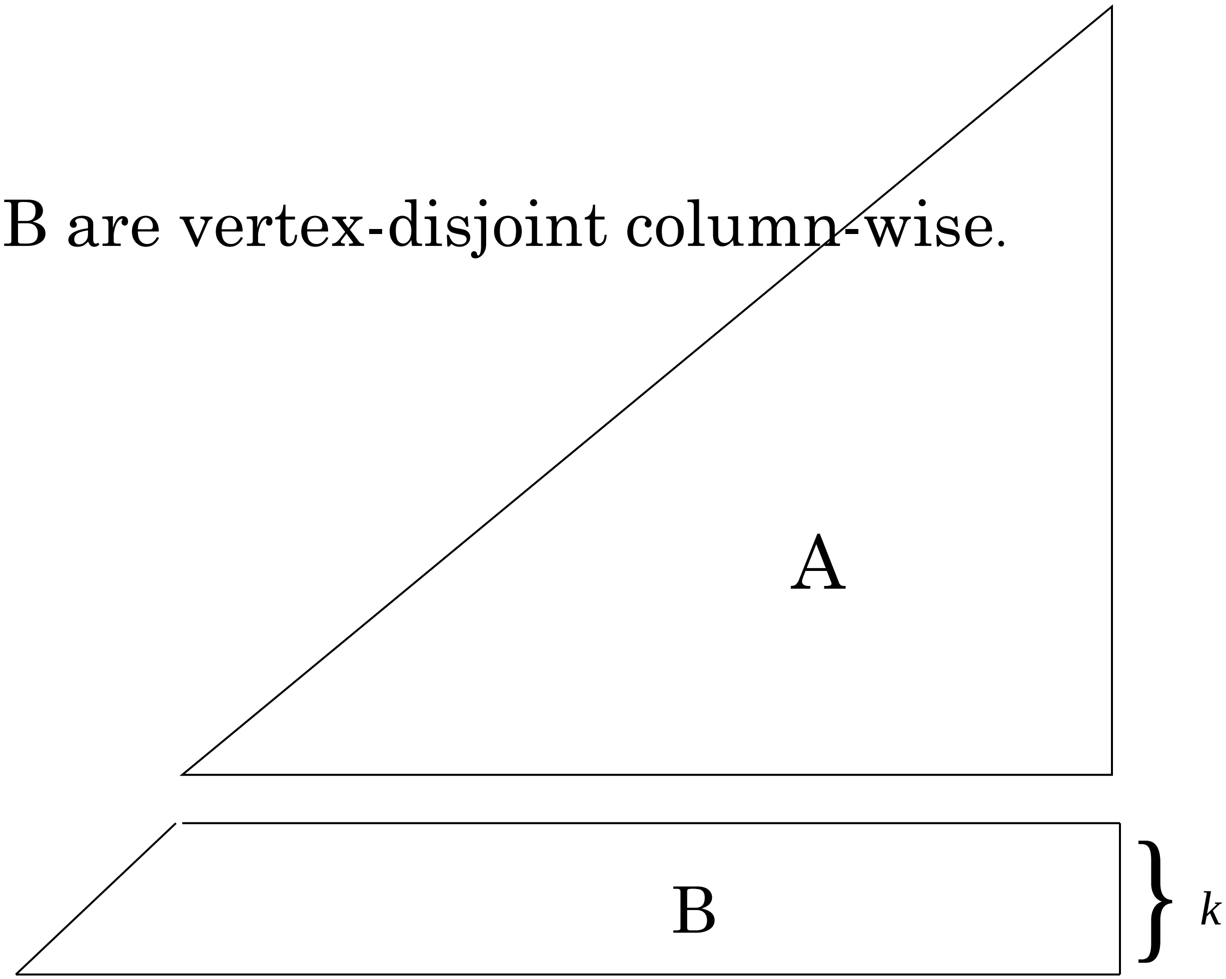
Pregnant star



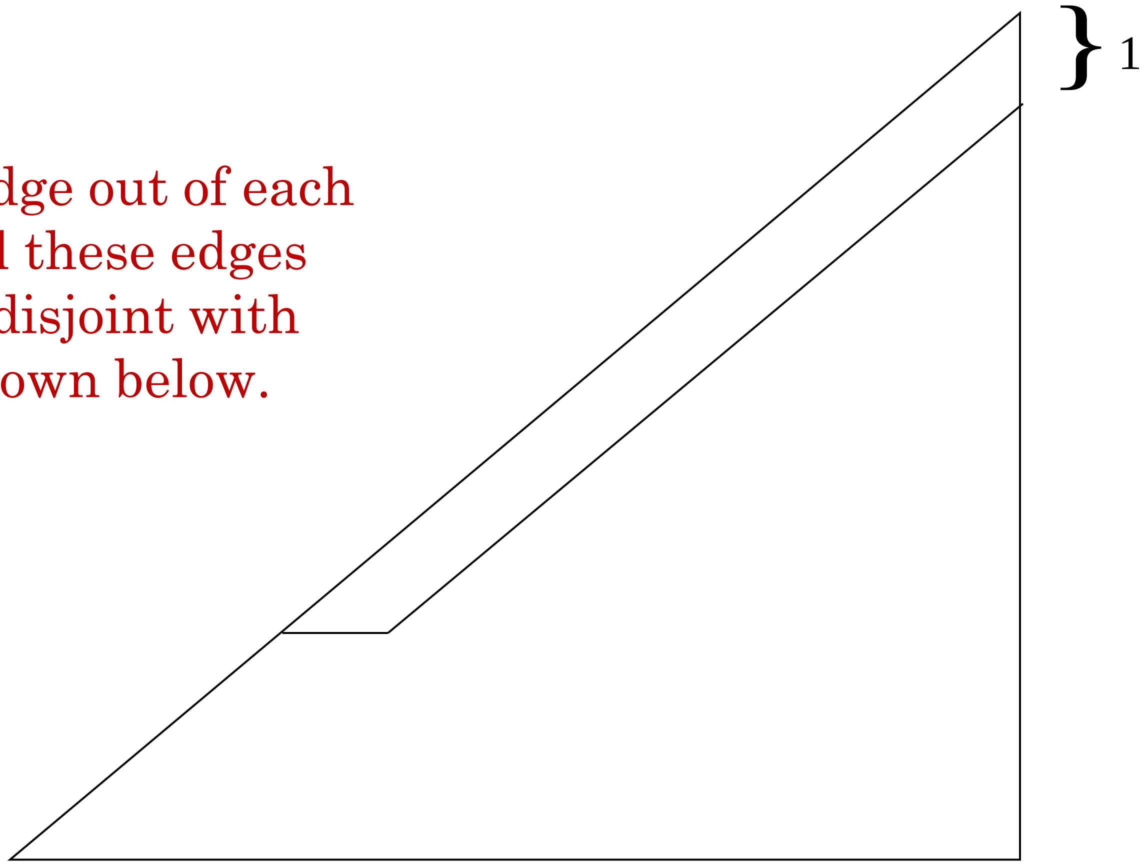
General idea

- Cut and Paste!
- This requires a good idea of decomposing the graphs into bigger parts first.
- If the graph has larger diameter, then using paths or path (linear) forests as its members.
- On the other hand if the graph is denser, then stars or star forests are good candidates.

A and B are vertex-disjoint column-wise.



- Take one edge out of each column and these edges are vertex-disjoint with the edges down below.



Final remarks

- I am starting to work on this problem again after 25 years.
- I believe that this problem can be solved in the near future.
- God bless the people who work on this problem!