Algebra (I) **Final** (two pages) 2014, 1, 13

1. Prove or disprove the following statements. (2 points each)

   (a) Let $G = \boldsymbol{R} \setminus \{-1\}$ and $x \circ y = x + y + x \cdot y$ for all x and y in G. Then $\langle G, \circ \rangle$ is a group. (***R*** is the set of all real numbers.)

   (b) $Z_3 \times Z_6$ is isomorphic to Z_{18}.

   (c) Let G be a group which acts on the set X with group action $*$. Then for each $y \in X$, $Gy = \{g \in G: g*y = y\}$ is a subgroup of G.

   (d) If R is an integral domain which has 101 elements, then R is a field.

   (e) Any two groups of order 4 (with 4 elements) are isomorphic.

   (f) If F is a finite field with 3 elements, then $x^3 + x + 1$ is irreducible over F.

   (g) If N is an idea of a field F and $N \neq \{0\}$, then N = F.

   (h) Let α be a rational number and φ_α be a mapping from Q[x] into Q defined by $\varphi_\alpha(f(x)) = f(\alpha)$ for each f(x) in Q[x]. Then φ_α is a ring homomorphism. (Q is the set of rational numbers.)

   (i) Let N be an ideal of Q[x] and $x^2 - 1$ be an element of N. Then N contains the element $x - 1$.

   (j) $X^5 + 5x^4 - 15x^3 + 30x^2 - 70$ is irreducible over Q.

2. Let $\phi(n)$ denote the number of positive integers not greater than n which are relatively prime with n. Prove that $< \boldsymbol{Z_n}, + >$ is a cyclic group which has $\phi(n)$ generators. (3 points)
3. Prove that the set of complex numbers with rational coefficients, $\{a + bi \mid a, b \in Q\}$, is a field. (3 points)
4. Prove that if G is a finite group of order m and H is a subgroup of G with n elements, then n is a divisor of m, i. e. $n|m$. (3 points)
5. Prove that if N is an ideal of Z (the set of integers), then N contains all integers which are multiples of a fixed integer. (3 points)
6. Find a ring which is not commutative. (3 points)
7. Prove that if R is a ring with unity and M is an ideal of R, then R/M is a field if and only if M is a maximal ideal of R. (5 points)