Abstract Algebra

Solutions of Quiz 4

1.

a) True. Every finite integral domain is a field.

Proof. Let $R = \{a_0 = 1, a_1, \dots, a_{100}\}$. For any $a_i \neq 0 \in R$, let ϕ_{a_i} be a map from R to a_iR defined by $\phi_{a_i}(a) = a_ia$, then ϕ_{a_i} is a bijection since if $a_ia_j = a_ia_k$ then $a_j = a_k$ (R is an integral domain, so we have cancelation law). Now $a_ia \in R$ since R is a ring, we have $a_iR = \{a_ia_0, a_ia_1, \dots, a_ia_{100}\} = R$, and hence $a_ia_j = 1$ for some j, therefore a_i is invertible, thus R is a field. □

b) Since in an integral domain, the polynomial $x^2 - 1$ has only two solutions 1, and -1. So for $a \in \mathbb{Z}_{17}\backslash\{0\}$, and $a \neq 1, p-1$, there exists an element $a^{-1} \neq a$, such that $aa^{-1} \equiv 1 \mod 17$, therefore $(16)! \equiv 1 * 16 \equiv -1 \mod 17$.

c) False. $\mathbb{R}$ is a field, but char$(\mathbb{R}) = 0$.

d) By Fermat's little theorem, we have if $(a, p) = 1$ then $a^{p-1} \equiv 1 \mod p$.

e) False. $\mathbb{R}[x]$ is not a division ring, since $x^{-1} \notin \mathbb{R}[x]$.

2. $Q_4 = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{Q}, i^2 = j^2 = k^2 = -1, ij = -ji = k\}$ is a noncommutative division ring. You should verify it is a ring, and it's easy to see it is noncommutative, and $1 \in Q_4$. We are going to show each nonzero element have inverse. For $a + bi + cj + dk \in Q_4$,

$$(a + bi + cj + dk)^{-1} = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}$$

.

3. It's easy to verify it is a commutative ring with unity (verify yourself). Now for $a + bi \in \{a + bi \mid a, b \in \mathbb{Q}\}$,

$$(a + bi)^{-1} = \frac{a - bi}{a^2 + b^2}.$$

Hence $\{a + bi \mid a, b \in \mathbb{Q}\}$ is a commutative division ring.

4. See the note week 13, page 4.