Abstract Algebra

Solutions of Quiz 3

1.

a) False: C may not a subgroup since it may not closed. (i.e., for $c, d \in C, cd \notin C$).

b) False: counterexample: Let $G = D_4 = \{\rho_0, \rho_1, \rho_2, \rho_3, \mu_1, \mu_2, \mu_3, \mu_4\}$ where $\rho_0 = e, \rho_1 = (1234), \rho_2 = (13)(24), \rho_3 = (1432), \mu_1 = (12)(34),$ $\mu_2 = (14)(23), \mu_3 = (13), \mu_4 = (24)$, let $H = \{\rho_0, \rho_2, \mu_1, \mu_2\}$, and $K = \{\rho_0, \mu_1\}$. Then $K \lhd H$ and $H \lhd G$ but $K \ntriangleleft G$.

c) Since the factor group of G over H is the set of all left cosets of H, hence $|G/H| = (G : H) = |G|/|H|$.

d)

$$e * g = ege^{-1} = g \text{ for all } g \in G$$

For all $g \in G,\ h_1, h_2 \in H$

$$(h_1h_2) * g = (h_1h_2)g(h_1h_2)^{-1} = h_1(h_2gh_2^{-1})h_1^{-1} = h_1 * (h_2 * g)$$

$\Rightarrow H$ act on G.

e)

(1) Closed: For $a, b \in G_y$, $(ab) * y = a * (b * y) = a * y = y$, we have $ab \in G_y$.

(2) Identity: $e \in G_y$ since $e * y = y$ by the definition of group action.

(3) Inverse: For $g \in G_y$, $g^{-1} * y = g^{-1} * (g * y) = e * y = y$, hence $g^{-1} \in G_y$.

$\Rightarrow G_y$ is a subgroup of G.

2. Example 1: $M_{2\times 2}(\mathbb{R})$ is a noncommutative ring. (You need to verify it is a ring). It is noncommutative since

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} = \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix} \neq \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

Example 2: Let

$$Q = \{a + b * i + c * j + d * k \mid a, b, c, d \in \mathbb{R}, i^2 = j^2 = k^2 = -1, ij = -ji = k\}$$

Then clearly, it is noncommutative since $ij = -ji$. (Verify it is a ring yourself).

3. (the proof is from the note week 10, page1) From 1.(e), we have $G_x \leq G$. Define a mapping from the set G_x into the collection of left cosets of G_x in G by

$$\psi(y) = g_1G_x \text{ where } g_1x = y.$$

(Note. Since Gx is an orbit contains $x, \forall y \in Gx, \exists g \in G$, s.t. $gx = y$.)
First, we claim ψ is well-defined. Suppose that $g_2x = y, g_2 \in G$. Then $g_1x = g_2x$. Hence $(g_2^{-1}g_1)x = x, g_2^{-1}g_1 \in G_x$. This implies that g_1G_x and g_2G_x are the same coset.
To conclude the proof, we claim that ψ is 1-1 and onto. Since $\psi(y) = \psi(y^{'})$, $g_1G_x = g_2G_x$ where $g_1x = y$ and $g_2x = y^{'}$. By $(g_2^{-1}g_1)x = x$, we have $y = y^{'}$.

Finally, let $g_1 G_x$ be a left coset. Clearly, $g_1 x \in Gx$ and $g_1 x = x_1 \in Gx$. By definition $\psi(x_1) = g_1 G_x$, hence ψ is 1-1 and onto. This implies that

$$|Gx| = (G : G_x)$$

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