Abstract Algebra

Solutions Quiz 2

1.

a) $\mathbb{Z}_3 \times \mathbb{Z}_6$ is NOT isomorphic to $\mathbb{Z}_{18}$ since $\mathbb{Z}_{18}$ has an element of order 18, but the elements in $\mathbb{Z}_3 \times \mathbb{Z}_6$ have orders at most 6.

b) Suppose $H \leq G$ and $(G : H) = 2$, then H has two left cosets (and two right cosets) in G. If $g \in H$, then $gH = H = Hg$. If $g \notin H$, then $gH = G - H = Hg$. That is $H \triangleleft G$. The index of the subgroup H is 2, thus it is normal in G.

c) No, there is a counterexample A_4. Suppose $H < A_4$ and $|H| = 6$, then $(G : H) = 2$ hence $H \triangleleft G$. It means for any $g \in G$, $g^{-1}Hg \in H$. Now $|H| = 6$ implies H contains a three cycle since the number of non-3-cycle element in A_4 is 4. W.L.O.G, suppose $(123) \in H$, then $(12)(34)(123)(12)(34) = (142) \in H$, $(13)(24)(123)(13)(24) = (341) \in H$, and $(14)(23)(123)(14)(23) = (432) \in H$, hence there are at least 7 elements $\{e, (123), (123)^{-1}, (142), (142)^{-1}, (134), (134)^{-1}\}$ contained in H, contradicting to $|H| = 6$.

d) $(12)(13)(16)(15)(14) = (145632) \neq (123654)$.

e) True. Let $H = \ker(\phi)$. We are going to show $\forall h \in H$, $a^{-1}ha \in H$. Now $\phi(a^{-1}ha) = \phi(a^{-1})\phi(h)\phi(a) = \phi(a^{-1})e^{'}\phi(a) = \phi(a^{-1})\phi(a) = \phi(a^{-1}a) = \phi(e) = e^{'}$ Thus $a^{-1}ha \in H$, hence $H \triangleleft G$.

2. $\phi_1 : (Z, +) \to (Z, +)$ defined by $\phi_1(x) = x$, then

$$\phi_1(x + y) = x + y = \phi_1(x) + \phi_1(y).$$

$\phi_2 : (Z, +) \to (Z, +)$ defined by $\phi_1(x) = 0$, then

$$\phi_1(x + y) = 0 = 0 + 0 = \phi_1(x) + \phi_1(y).$$

$\phi_3 : (Z, +) \to (R, \cdot)$ defined by $\phi_3(x) = e^x$, then

$$\phi_3(x + y) = e^{x+y} = e^x \cdot e^y = \phi_3(x) + \phi_3(y).$$

3. First, we are going to show for any $g \in G$, $|gH| = |H|$. Let $\phi_g : H \to gH$ defined by $\phi_g(h) = gh$, then (1) for any $gh \in gH$, we have $\phi_g(h) = gh$, hence ϕ_g is onto; (2)For $\phi_g(h_1) = \phi_g(h_2)$, we have $gh_1 = gh_2$, which implies $h_1 = h_2$, thus ϕ_g is one-to-one. By(1)(2), we have $|H| = |gH|$.
Second, we are going to show for any $g_1, g_2 \in G$, either $g_1H = g_2H$ or $g_1H \cap g_2H = \phi$. Suppose $g_1H \cap g_2H \neq \phi$, let $c \in g_1H \cap g_2H$, then $c = g_1h_1 = g_2h_2$ for some $h_1, h_2 \in H$. Now $g_1h_1 = g_2h_2$ implies $g_1 = g_2h_2h_1^{-1} \in g_2H$, hence $g_1H = g_2H$.
The union of all cosets of H is G, hence the cosets of H partition G into same cardinality parts, thus $|H|||G|$.