前言

如果說圖論(Graph Theory)是二十一世紀最具影響力的數學，一定有不少的「數學家」會有不同的意見，因為「它」看起來不像；不論是敘述、例子，或是定理看起來就是不難，例如四色定理，中小學生都很清楚它在說什麼，也許很少人確實知道如何證明這個定理，那也影響不大，看起來不難。

在二十一世紀，每個人都會同意手機的使用，改變了人類生活的方式，尤其是智慧型手機的廣泛使用以及它令人難以置信的多方位功能，改變仍會繼續下去。是什麼樣的數學在扮演近代科技發展的推手呢？是Queen of Mathematics數論嗎？是The most beautiful mathematics幾何嗎？還是所存在連續事物的基礎角色微積分呢？

離散數學(Discrete Mathematics)無庸置疑是近代「數位化」過程中最具影響力的數學；本來一切都是「離散」的事物；而圖論絶對是具有最多應用的專題，我不可能用短短的篇幅來描述它所有的成就，總之，少了它，We go nowhere。

在這總體的學問中，我們很難在課堂上把全部相關的專題都介紹；因為，除了內容廣泛之外，在介紹的過程中，也許又有了一個分枝正在慢慢地茁壯發芽、開花、結果哩！以下，所有的焦點都放在圖的分割(Graph Decomposition)上面，希望我們可以從這裡面發現更多的內容以及做出一份貢獻，尤其在尋求未解決問題的答案上面，有所作為。

以下所提及的內容，都是我個人認為適合介紹的課題，也許有很多其他學者認為重要的內容沒有被提出來說明，這並不表示那些研究比較不重要，而是我們所要學習的內容太廣泛以至於只有部分概念出現於這學期所涵蓋的範圍；內容如有瑕疵，全部都是筆者的責任。無論如何，希望大家能有一些收穫，尤其在了解圖分割方面有十分具體的認識。

簡介

圖分割(Graph decomposition)有數個不一樣的定義，從Wiki上面不難搜尋到它們；我們所討論的分割基本上是在分割一個圖G的邊集合(Partition the edge set of G)，在分成多個邊集合之後，再看看這些集合所生成的圖的形式(成員，Member)。所以，討論的對象分別是母圖G(Host Graph)以及所分割出來的子圖(Member)究竟是哪一類的圖；例如，分割出來的圖都是star，則此分割也稱為是Star Decomposition。

最古典的這類研究，不外乎史坦那三元系(Steiner Triple System)，它對應於探討那一類的完全圖K_v可以分成全部都是三角形K_3的問題；這問題的解決已經是兩百五十年前的事了。但是，最早與分割相關的研究就要再早一百年了；歐拉(Euler)在證明尤拉圖存在的充要條件為每個點的度數都是偶數而且該圖為連通圖的同時也已經隱含了一個簡單偶圖(Simple even graph)，都可以分割成大小不一定相同的圈(Cycle)。

分割的問題會一直被關注下去，因為，除了它本身引申出的諸多難題(Open problem, Conjectures)，它的應用也是不斷的被發現；尤其是在特殊傳輸網路的設計上，包括OOC(Orthogonal optical code)，CAC(Conflict-avoiding code)，...；還有群試理論以及分散式密碼系統(Secret sharing scheme) 的資訊訊息比值的研究等等。

在這個講義中，我們將盡可能地去介紹所有相關的課題；只是要涵蓋所有的研究是一件不可能的任務。所以，幾個主題的選擇都是以我個人曾經做過或參與過的研究為主軸，再搭配一些研究工作的搜尋(Survey)以充實內容，希望這能夠作為對這專題有興趣的同學或研究人員一個有幫助的參考資料。

【後記】由於內容的呈現大都以英文較為方便；因此，這份講義也都以英文書寫，不過比較細節的計算與說明會適時地加註中文。

Chapter 1 Decomposition of Complete Graphs

Graph decomposition deals with a partition of the edge set of a graph such that each subset satisfies a constraint we assigned before hand. When considering a subset we in fact consider the subgraph "edge-induced" by the subset. We remark here that the subgraph is edge-induced not induced. The difference between them is not difficult to see since the later one is obtained by using a vertex subset to induce a subgraph instead of using the edge set.

We shall call the subgraphs obtained in a decomposition "the members of the decomposition". For convenience, if a graph G can be decomposed into members $G_1, G_2, ..., G_t$, then we represent this decomposition by "$G = G_1 + G_2 + \cdots + G_t$. In case that all members are mutually isomorphic, then G can be written as $G = t \cdot \tilde{G}$ where $\tilde{G} \cong G_i$, for $i = 1, 2, ..., t$. Talking about the decomposability of G into $\tilde{G}$'s, we shall use $\tilde{G}|G$ to denote that G can be decomposed into copies of G.

In what follows, we study the decomposability of a complete graph K_n into copies of $\tilde{G}$.

Lemma 1.1. If $\tilde{G}|K_n$, then

1. $|\tilde{G}| \leq n$,
2. $\|\tilde{G}\| | \binom{n}{2}$ and
3. $g.c.d.\{d_{\tilde{G}}(v), v \in v(\tilde{G})\}|n-1$.

Proof. The first two necessary conditions are easy to see. We claim the third one. Since $\tilde{G}|G$, $n-1 = \sum_{v \in S} deg_{\tilde{G}}(v)$ where $S \subseteq V(\tilde{G})$.(S is a multi-set.) Let $d = g.c.d.\{d_{\tilde{G}}(v), v \in v(\tilde{G})\}$. Then $n-1 = \sum_{v \in S} d \cdot \frac{deg_{\tilde{G}}(v)}{d} = d\left(\sum_{v \in S} \frac{deg_{\tilde{G}}(v)}{d}\right) = d \cdot x$. By the fact that x is an integer, we have the proof. ■

Corollary 1.2. If $\tilde{G}$ is an r-regular graph, then $r|n-1$ provided $\tilde{G}|K_n$.

Theorem 1.3. K_n can be decomposed into 1-factors if and only if n is even.

Proof. Since a 1-factor is a 1-regular spanning subgraph of K_n, n must be even. On the other hand, if n is even, then the set of $n-1$ 1-factors can be obtained by the following construction. First, we construct a regular convex $(n-1)$-gon define on $\{1, 2, ..., n-1\}$ (see Figure 1.1.). Let 0 be the center(point). Now, the first 1-factor is obtained by line segment $\{0, 1\}$ and all the other segments which are perpendicular to $\{0, 1\}$. Similarly, we can obtain the other 1-factors by using $\{0, 2\}, \{0, 3\}, ..., \{0, n-1\}$ respectively. This completes the proof.

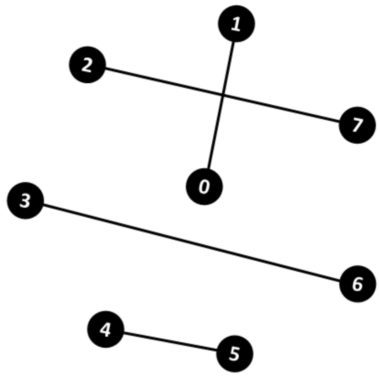


Figure 1.1. 7-gon

■

Definition 1.4. (1-factorization) A 1-factorization of a graph G is a decomposition of G into 1-factors. Clearly, if n is even, then K_n has a 1-factorization. In fact, there are quite a few different 1-factorizations of a labelled K_n. (This is not difficult to see.)

- If G has a 1-factorization, then G must be a regular graph.
- Not all regular graph of even even order has a 1-factorization, e.g, Petersen graph has no 1-factorizations.

Definition 1.5. (Near 1-factor) In a graph G of odd order, a near 1-factor is a 1-regular subgraph of order $|G| - 1$, i.e., a near 1-factor is incident to all(but one vertex) vertices of G.

Theorem 1.6. K_n can be decomposed into near 1-factors if and only if n is odd.

Proof. Delete one vertex from a 1-factorization of K_{n+1}. ■

把一個圖分割成配對(Matchings)最知名而且有效率的方法是利用邊著色的概念。

Definition 1.61. A k-edge-coloring of a graph G is a mapping $\varphi : E(G) \to \{1, 2, ..., k\}$ such that incident edges receive distinct image, i.e., $\forall e, f \in E(G), \varphi(e) \neq \varphi(f)$ whenever $e \cap f \neq \phi$.

A graph which admits a k-edge-coloring is called k-edge-colorable.

Definition 1.6.2 $\chi'(G) = \min\{k|G$ is $k -$ egde-colorable$\}$ is the chromatic index of G. Clearly, if G is k-edge-colorable, then G is k'-edge-colorable for each $k' \geq k$. The following results are easy to see.

Fact a. $\chi'(G) \geq \Delta(G)$ where $\Delta(G)$ is the maximum degree of G.

Fact b. $\chi'(K_n) = \Delta(K_n) = n - 1$ if and only if n is even.

Fact c. $\chi'(K_{2m+1}) = 2m + 1$.

Definition 1.6.3. A graph G is said to be overfull if $\Delta(G) \cdot \lfloor \frac{|G|}{2} \rfloor < \|G\|$.
Clearly, this happens only if $|G|$ is odd.

Fact d. If G is overfull, then $\chi'(G) > \Delta(G)$.

Fact e. If G is the Petersen graph, then $\chi'(G) = 4$.

Fact f. If G is a cycle, then $\chi'(G) = 2$ if and only if G is an even cycle.

Theorem 1.6.4. (Vizing) If G is a simple graph, then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Theorem 1.6.5. (Vizing) If G is a multi-graph with multiplicity α, then $\chi'(G) \leq$

$\Delta(G)+\alpha$. (α:重邊數)

以上兩個定理的證明都用到所謂的Fan-Sequence，請參卓相關資料以取得證明。

Definition 1.6.6. If G is a simple graph, then G is of Class 1(respectively Class 2) if $\chi'(G)=\Delta(G)$(respectively $\chi'(G)=\Delta(G)+1$)

Theorem 1.6.7. (Random graphs) Almost all graphs are of Class 1.

Theorem 1.6.8. (König's Theorem) If G is a bipartite graph, then G is of Class 1.

Theorem 1.6.9. (Equitable-edge-coloring) If G has a k-edge-coloring φ, then G has a k-edge-coloring π such that $\forall i,j \in \{1,2,...,k\}$, $||\pi^{-1}(i)|-|\pi^{-1}(j)|| \leq 1$. Subsequently, if $\frac{\|G\|}{k}$ is an integer, then $|\pi^{-1}(i)|=|\pi^{-1}(j)|$.

Proof. The proof follows by consider two classes of $\varphi^{-1}(i)$ and $\varphi^{-1}(j)$ and make them equitably. (?) ■

以下，我們探討其他形式的分割。

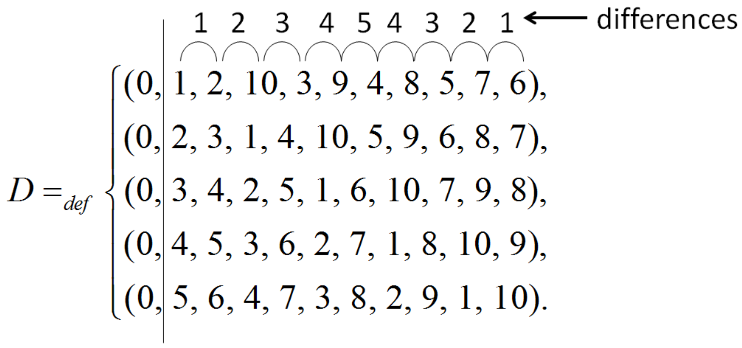
Theorem 1.7. K_n can be decomposed into Hamiltonian cycles if and only if n is odd and $n \geq 3$.

Proof. Since a Hamiltonian cycle is a 2-regular graph, $2|n-1$. Hence n is odd. On the other hand, we need to provide a decomposition of K_n for each odd $n \geq 3$. Let $V(K_n)=\{0,1,2,...,n-1\}$. Let B_0 be the cycle defined as follows and then the other cycles are obtained accordingly.

$$B_0 = \left(0, 1, 2, n-1, 3, n-2, ..., \frac{n-1}{2}, \frac{n+3}{2}, \frac{n+1}{2}\right),$$
$$B_1 = \left(0, 2, 3, 1, 4, n-1, ..., \frac{n+1}{2}, \frac{n+5}{2}, \frac{n+3}{2}\right),$$
$$\vdots$$
$$B_{\frac{n-3}{2}} = \left(0, \frac{n-1}{2}, \frac{n+1}{2}, \frac{n-3}{2}, \frac{n+3}{2}, ..., \frac{2n-4}{2}, n-1\right).$$

In fact, $B_i = B_0 + (i-1)$ where $(n-1)+1 =_{def.} 1$. ■

For clearness, we use $n = 11$ to depict the decomposition.



(Note) The above decomposition is also known as Walecki's decomposition. The proof for correctness will be obtained by checking the differences(half) between all vertices.

Definition 1.8. (Difference) Let X be a nonempty subset of Z_n. Then the difference set ΔX of X is defined as follows: $\Delta X = \{x - y | x, y \in X \text{ and } x \neq y\}$.

Example $X = \{1, 2, 4\}, \Delta X = \{\pm 1, \pm 2, \pm 3\}$.

Definition 1.9. (Half-difference) $\Delta_2 X = \{\min\{x - y, n - (x - y)\} | x, y \in X \text{ and } x > y\}$.

Example $X = \{1, 3, 9\} \subseteq Z_{11}, \Delta_2 X = \{2, 3, 5\}$. Note that in $\Delta_2 X$ contains only positive integers not greater than $\frac{n}{2}$ if $X \subseteq Z_n$. $\Delta_2 X$ is also known as "circular" difference set of X.

(Fact) If $|X| = k$, then $|\Delta X| \le k(k-1)$ and $|\Delta_2 X| \le \binom{k}{2}$.

Definition 1.9. (Circulant graph) Let $D \subseteq \{1, 2, ..., \lfloor \frac{n}{2} \rfloor\}$. Then the graph $G_n(D)$ is a circulant subgraph of K_n such that $V(G_n(D)) = V(K_n)$ and $E(G_n(D)) = \{\{u, v\} | u, v \in Z_n$ and $|u - v|_2 \in D\}$.

Fact 1 If n is even, then $G_n\left(\{\frac{n}{2}\}\right) = G_n\left(\frac{n}{2}\right)$ is a 1-factor.

Fact 2 If n is odd, then $G_n(D)$ is a $2k$-regular graph where $k = |D|$.

Note. $G_n(D)$ is also a $2k$-factor of K_n since $G_n(D)$ is a spanning $2k$-regular subgraph of K_n.

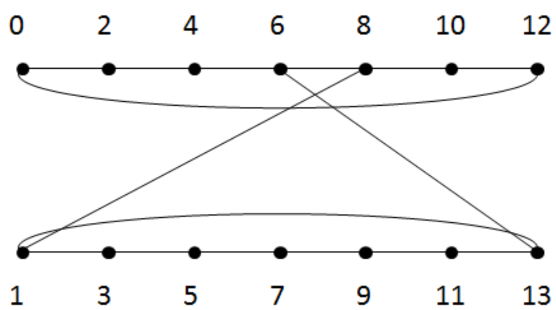
Fact 3 If n is even, then $G_n\left(\{2, \frac{n}{2}\}\right)$ can be decomposed into 3 1-factors.

Proof. First, if $\frac{n}{2}$ is even, then $n = 4m$. This implies that $G_n(\{2\})$ is a disjoint union of two even cycles. So, the proof follows by choosing two 1-factors from $G_n(2)$ and $G_2\left(\frac{n}{2}\right)$ is itself a 1-factor. On the other hand, if $\frac{n}{2}$ is odd, i.e., $n = 4m + 2$. For clearness, we look at the following example and catch the idea.

$n = 14$

$$G_{14}(2) = (0, 2, 4, 6, 8, 10, 12) \cup (1, 3, 5, 7, 9, 11, 13)$$

$$G_{14}(7) = (0, 7) + (1, 8) + (2, 9) + (3, 10) + (4, 11) + (5, 12) + (6, 13)$$



$$(0, 2, 4, 6, 13, 11, 9, 7, 5, 3, 1, 8, 10, 12)$$

$$+ (0, 7) + (2, 9) + (3, 10) + (4, 11) + (5, 12) + (6, 8) + (1, 13)$$

Hence, $G_{14}(\{2,7\})$ can be decomposed into a Hamilton cycle and a 1-factor.

General case:

$$\begin{aligned}
G_n(2) &= (0,2,4,...,2m,2m+2,...,4m) \cup (1,3,5,...,2m+1,2m+3,...,4m+1) \\
&+ (0,2m+1) + (1,2m+2) + \cdots + (2m,4m+1) \\
&= (0,2,4,...,2m,4m+1,4m-1,...,1,2m+2,...,4m) \\
&+ (0,2m+1) + \mathbf{(1,\ 2m{+}2)} + \cdots + \mathbf{(2m,\ 4m{+}1)} + \cdots + (2m-1,4m) \\
&+ (2m,2m+2) + (1,4m+1).
\end{aligned}$$

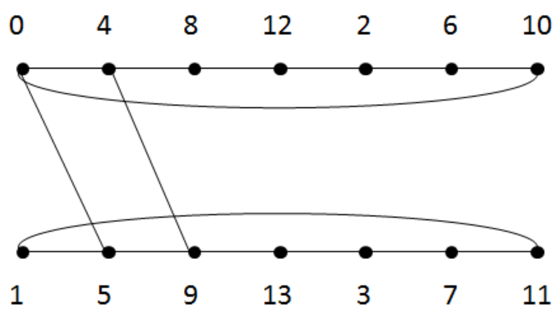
■

Fact 4 If n is even, then for each $2 \le k < \frac{n}{2}, G(k-1,k)$ can be decomposed into 4 1-factors.

Proof. (Exercise).

Example $n = 14, k = 5,$

$$G(4) = (0,4,8,12,2,6,10) \cup (1,5,9,13,3,7,11)$$
$$G(5) = (0,5,10,1,6,11,2,7,12,3,8,13,4,9)$$

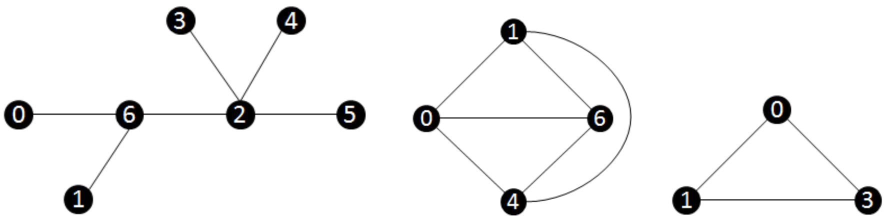


$$(0,5,1,11,7,3,13,9,4,8,12,2,6,10) + (10,1,6,11,2,7,12,3,8,13,4,9,0,5)$$

(*) $G(\{4,5\})$ is a union of two Hamilton cycles of even length and therefore a union of 4 1-factors.

Definition 1.10. (Graceful labeling or β-labeling) A graceful labeling is a mapping $\varphi : V(G) \to \{0, 1, 2, ..., \|G\|\}$ such that $\{|\varphi(u) - \varphi(v)| : u, v \in E(G)\} = \{1, 2, 3, ..., \|G\|\}$. Here, $|\varphi(u) - \varphi(v)|$ is called the weight of uv. A graph is graceful if it has a graceful labeling.

Examples



(Fact 1) K_n has a graceful labeling if and only if $n \leq 4$.

(Fact 2) Any caterpillar has a graceful labeling.

Conjecture 1.11. Every tree has a graceful labeling.

Theorem 1.12. If G is a graceful graph of size k, then K_{2k+1} can be decomposed into $2k+1$ copies of G.

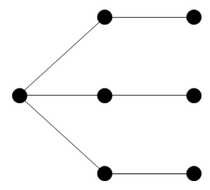
Proof. The proof follows by cyclically permuting the set of vertices. ■

Definition 1.13 (α-labeling) A β-labeling φ of G is an α-labeling, if G has a bipartition of $V(G)$, (A, B), such that $\max\{\varphi(a) : a \in A\} < \min\{\varphi(b) : b \in B\}$.

(Fact 3) In order to have an α-labeling(of G), G has to be bipartite.

(Fact 4) Not all trees have an α-labeling. (?)

Example



Graph with no α-labeling

Theorem 1.14. If G has an α-labeling then K_{2pk+1} can be decomposed into copies of G where $\|G\| = k$ and p is a positive integer.

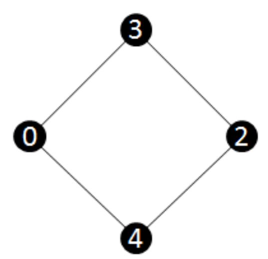
Proof. We need p base graphs. In fact, instead of using G to generate K_{2pk+1}, we use p copies of G. Let $\tilde{G}$ be the disjoint union of p copies of G. Since G has an α-labeling φ, we let $G_i = (A_i, B_i)$, $i = 1, 2, ..., p$ and G_1 has the same labeling as that of G, φ. Hence, if $A_1 = \{a_1^{(1)}, a_2^{(1)}, ..., a_s^{(1)}\}$ and $B_1 = \{b_1^{(1)}, b_2^{(1)}, ..., b_t^{(1)}\}$ and $\max \varphi(A_1) < \min \varphi(B_1)$, then we define $\tilde{\varphi}$ of $\tilde{G}$ as follows:

$$\varphi(A_1) = \varphi(A_2) = \cdots = \varphi(A_p) \text{ and}$$
$$\varphi(B_i) = \varphi(B_1) + (i-1)p.$$

This implies that $\Delta(V(G_1)) = \{1, 2, ..., k\}$, $\Delta(V(G_2)) = \{k+1, ..., k+k\}$, ..., $\Delta(V(G_p)) = \{(p-1)k+1, ..., pk\}$. Now, by Theorem 1.12, we conclude the proof.

Theorem 1.15. $C_4 | K_n$ if and only if $n \equiv 1 (\text{mod } 8)$.

Proof. ($\Rightarrow$) Since K_n can be decomposed into 4-cycles, n must be odd and $4 | \binom{n}{2}$. This implies that $n \equiv 1 (\text{mod } 8)$. ($\Leftarrow$) By Theorem 1.14., it suffices to show that C_4 has an α-labeling in K_9, and this is true by the following labeling of C_4.



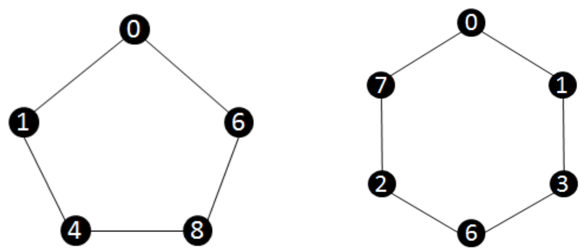
■

(Note) $C_4|K_n$ is also known as a 4-cycle system of order n.

(Note) We shall decompose K_n into C_4's by using a different method in later section.

Definition 1.16. A ρ-labeling of a graph G based on $[1, n]$. is a labeling $\varphi : V(G) \to [0, n-1]$ such that the set of weights of G is exactly $\{1, 2, ..., \|G\|\}$.

Example C_5 has a ρ-labeling based on $[0, 10]$, and C_6 has a ρ-labeling on [0, 12].



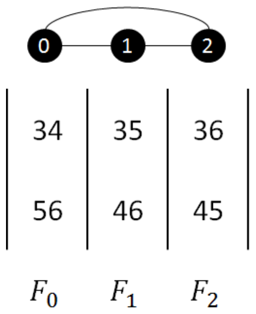
Conjecture 1.17. Every tree of size k has a ρ-labeling based on $[0, 2k]$.

Conjecture 1.18. Let T be a tree of size k. Then $T|K_{2k+1}$. Note here that cyclical decomposition is not the only way to decompose the complete graphs into isomorphic subgraphs. In what follows, we introduce the recursive constructions.

Theorem 1.19. If $K_3|K_v$, then $K_3|K_{2v+1}$.

Proof. Since $K_3|K_v$, $v \equiv 1$ or $3(\mod 6)$. Let $\mathfrak{F} = \{F_0, F_1, ..., F_{v-1}\}$ be a 1-factorization of K_{v+1} which is defined on $[v, 2v]$. Moreover, let the K_v mentioned above be defined on $[0, v-1]$. Now, consider K_{2v+1} defined on $[0, 2v]$. Let $\mathbb{B}$ be the collection of K_3's defined as $\mathbb{B} = \mathbb{B}_1 \cup \mathbb{B}_2$ where $\mathbb{B}_1$ is the collection of K_3's in the decomposition of K_v and $\mathbb{B}_2 = \bigcup_{i=0}^{v-1}(i, F_i)$ in which (i, F_i) in the collection of K_3's $\{i, a, b\}$ where $\{a, b\}$ is an edge in F_i. ■

Example



$\Rightarrow$ $\{0,1,2\}$,$\{0,3,4\}$, $\{0,5,6\}$,
$\{1,3,5\}$,$\{1,4,6\}$, $\{2,3,6\}$, $\{2,4,5\}$.

Theorem 1.20. If $K_3|K_v$, then $K_3|K_{2v+7}$.

Proof. If $v = 1$, then clearly $K_3|K_9$. Assume that $v \geq 3$ and consider K_{v+7}. It suffices to prove that there are v 1-factors $\{F_1, F_2, ..., F_v\}$ in K_{v+7} such that after deleting there v 1-factors, the remaining subgraph of K_{v+7} can be decomposed into K_3's, see Figure 1. Let K_{v+7} be defined on Z_{v+7} and $n = v + 7$. Then $G_n(\{1, 3, 4\})$ is a disjoint union of K_3's. Now, let $G = K_{v+7} - G_n(\{1, 3, 4\})$. So, $G = G_n(\{2, 5, 6, ..., \frac{n}{2}\})$. It's left to show that $G_n(\{5, 6, ..., \frac{n}{2} - 1\})$ can be decomposed into $v - 3$ 1-factors(by Fact 3).

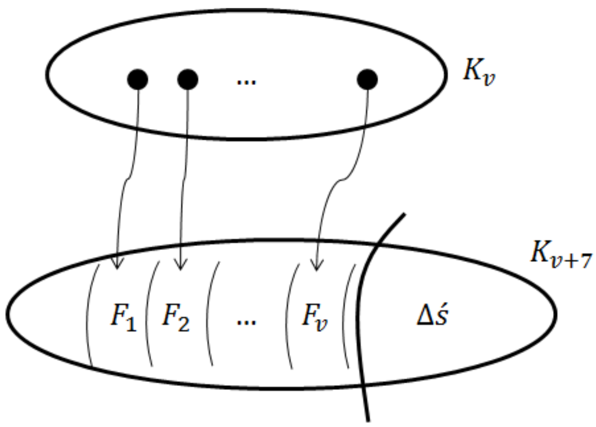


Figure 1. Depiction of the construction

First, if $\frac{n}{2} - 1$ is even, then by Fact 4, the graph can decomposed into $v - 3$ 1-factors since there are $\frac{v-3}{2}$ differences in $\{5, 6, ..., \frac{v+5}{2}\}$ and $\frac{v-3}{2} = \frac{n}{2} - 5$ is even. On the other hand, if $\frac{n}{2} - 1$ is odd, then $G_n(\frac{n-2}{2})$ is a Hamilton cycle of length n. This implies that G can be decomposed into $v - 3$ 1-factors.(?) ∎

(Note) $n = v+7, \frac{n-2}{2} = \frac{v+5}{2}, \gcd(n, \frac{v+5}{2})|(v+7) - 2 \cdot \frac{v+5}{2} = 2$. But, $\frac{v+5}{2}$ is odd, $\gcd(n, \frac{v+5}{2}) = 1$.

Fact 5 If $\gcd(n', n) = 1$, then $G_n(n')$ is a Hamilton cycle in K_n.

說明 遞迴建構(分割)的方法是非常常見的技巧，在其他領域如組合設計、特殊碼(code)的建造都扮演重要角色。我們當然也可以用直接建構的方法，只是較不容易。以下是兩個例子，它們分別需要比較特殊型式的拉丁方陣來協助建構。

Before we present the direct constructions for $K_3|K_v$, we remark that by using Theorem 1.19., Theorem 1.20.,$K_3|K_7$ and $K_3|K_9$, we conclude

Theorem 1.21. $K_3|K_v$ if and only if $v \equiv 1$ or $3 \pmod 6$.

The above theorem has been proved by T.P. Kirkman at 1847. In what follows, we are aiming at proving Theorem 1.21.by way of direct constructions.

Definition 1.22. (Idempotent commutative Latin square) A Latin square $L = [l_{i,j}]$ is idempotent if $l_{i,i} = i$ and L is commutative if $l_{i,j} = l_{j,i}$ for all i, j.

Example

1	4	2	5	3
4	2	5	3	1
2	5	3	1	4
5	3	1	4	2
3	1	4	2	5

Idempotent commutative Latin square

Theorem 1.23. An idempotent commutative Latin square of order n exists if and only if n is odd.

Proof. Let n be an odd integer and $L = [l_{i,j}]_{n\times n}$ where $l_{i,j} = i + j (\text{mod } n)$ by using $1, 2, ..., n$. Then L is a diagonal commutative Latin square of order n. Hence, an idempotent commutative Latin square of order n can be obtained by permutating the elements(entries) of L. (See example below.)

	1	2	3	4	5
1	2	3	4	5	1
2	3	4	5	1	2
3	4	5	1	2	3
4	5	1	2	3	4
5	1	2	3	4	5

$\Rightarrow$ Idempotent $\begin{pmatrix} 2 & 4 & 1 & 3 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

On the other hand, if L is an idempotent commutative Latin square of order n, then each element from $\{1, 2, ..., n\}$ an odd number of times, one at diagonal and even times outside of diagonal since it is commutative. ■

Definition 1.24. A commutative Latin square of order $2h$ with h 2×2 holes ia a commutative Latin square of order $2h$ such that along its diagonal there are h 2×2 Latin subsquares defined on $\{0, 1\}, \{2, 3\}, ..., \{2h - 3, 2h - 1\}$ respectively.

Example

0	1	4	5	2	3
1	0	5	4	3	2
4	5	2	3	0	1
5	4	3	2	1	0
2	3	0	1	4	5
3	2	1	0	5	4

Theorem 1.25. For each $h \geq 3$, a commutative Latin squares of order $2h$ with h 2×2 holes exists.

Proof. If h is odd, then the proof follows by blowing up each entry of an idempotent commutative Latin square of order h into a 2×2 Latin square respectively. But, if h is even, then it takes longer content to explain the existence, we omit the details. (Can you prove it?) ■

Now, we are ready to provide a direct construction to show that $K_3|K_v$ for each $v \equiv 1$ or $3 \pmod 6$.

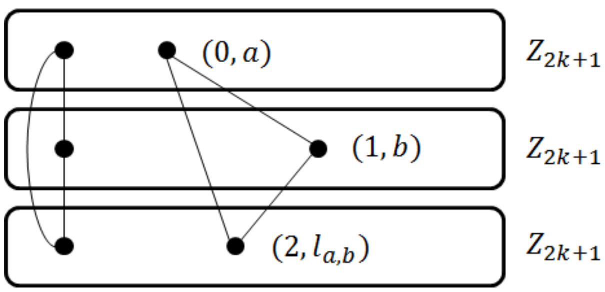
Another proof of Theorem 1.21.(Direct constructions)

Case 1. $v = 6k + 3$.

Let $L = [l_{i,j}]$ be an idempotent commutative Latin square of order $2k + 1$(defined on Z_{2k+1}). Let $V(K_v) = \{(x, y)|x \in Z_3 \text{ and } y \in Z_{2k+1}\}$. Now, we construct the decomposition directly by letting

(1) for each $y \in Z_{2k+1}$, $\{(0, y), (1, y), (2, y)\}$ defines a K_3; and

(2) for each $i \in Z_3$, $\{(i, a), (i + 1, b), (i + 2, l_{a,b})\}$ defines a K_3 where the first component modules 3 and $a \neq b$.



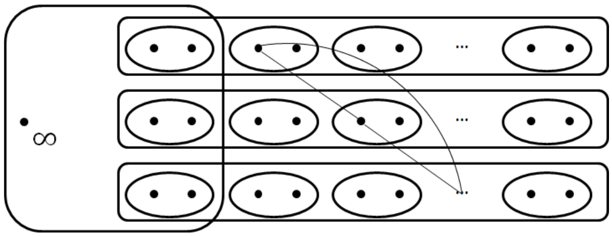
By direct checking, we obtain a K_3-decomposition of K_v.

Case 2. $v = 6k + 1$, $k \geq 3$. ($K_3|K_7$ and $K_3|K_{13}$(?) are easy to see.)

Let $L' = [l'_{i,j}]$ be a commutative Latin square of order $2k$ with k 2×2 holes. Now, let $V(K_v) = \{\infty\} \cup \{(x, y)|x \in Z_3 \text{ and } y \in Z_{2k}\}$. The decomposition of K_v can be obtained as follows:

(1) for each set $\{\infty, (i,j), (i,j+1) | i \in Z_3, j$ is a fixed even integer in $Z_{2k}\}$ we define seven K_3's; and

(2) for any two integers a and b not in the hole at the same time, let $\{(i,a), (i,b), (i+1, l'_{a,b})\}$ be a K_3 where the first component modules 3.



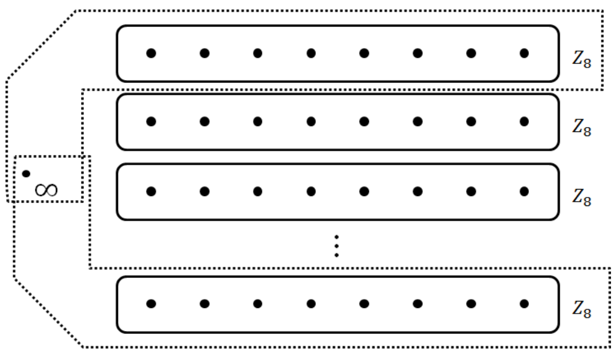
Note that we may count the number of K_3's in the decomposition and check every edge and see if it is in a K_3 we construct.

By a similar idea, we can prove $C_4 | K_v$ if and only if $v \equiv 1 (\text{mod } 8)$. As mentioned earlier the necessary condition is easy to see, so we show the direct construction.

Direct construction of $C_4 | K_{8k+1}$. Let $V(K_{8k+1}) = \{\infty\} \cup \{(i,j) | i \in Z_k$ and $j \in Z_8\}$. Then the collection of 4-cycles are in

(1) $\{\infty, (i, Z_8)\}$ for each fixed $i \in Z_k$; and

(2) for any two $i \neq j \in Z_k$, the set of C_4's $K_{8,8}$.



■

We note here that the following theorem is well-known and plays a very important role in graph decomposition.(You may prove it as an exercise.)

Theorem 1.26.(D. Satteau) Let m, n, k be three positive even integers such that $k \leq n \leq m$ and $2k|mn$. Then $K_{m,n}$ can be decomposed into C_{2k}'s, i.e., $C_{2k|K_{m,n}}$.

$C_4|K_{8,8}$ is just a special case of the above theorem.

把完全圖分割成圈的形式一直都是圖分割的研究主題，最著名的定理不外乎是由B. Alspach所主導的工作。

Theorem 1.27. (Cycle decomposition) If n is odd, $n \geq k \geq 3$ and $k|\binom{n}{2}$, then $C_k|K_n$, and if n is even, $n \geq k \geq 3$ and $k|\binom{n}{2} - \frac{n}{2}$, then $C_k|K_n - F$ where F is a 1-factor of K_n. Note that the proof of the above theorem occurred in several papers, we omit the detail here. B. Alspach also proposed an open problem about cycle decomposition in 1980 and this problem is claimed to be solved by several Australarian researchers which include D. Bryant.

Alspach's problem Let n be an odd integer (respectively even) $n \geq m_1 \geq m_2 \geq \cdots \geq m_t \geq 3$ and $\sum_{i=1}^{t} m_i = \binom{n}{2}$ (respectively $\sum_{i=1}^{t} m_i = \binom{n}{2} - \frac{n}{2}$). Then K_n(respectively $K_n - F$) can be decomposed into t cycles $Z_1, Z_2, ..., Z_t$ such that $\|Z_i\| = m_i, i = 1, 2, ..., t$. If we replace cycles by paths, and the longest path is of length $n - 1$, then the analog of Alspach's problem can be stated as follows:

Theorem 1.28. (Bryant et al.) Let $n - 1 \geq m_1 \geq m_2 \geq \cdots \geq m_t \geq 1$ such that $\sum_{i=1}^{t} m_i = \binom{n}{2}$. Then K_n can be decomposed into paths $Z_1, Z_2, ..., Z_t$ such that $\|Z_i\| = m_i, i = 1, 2, ..., t$.
Remark here that if n is odd, then the proof comparatively simpler. The idea is that we can use the Walecki's decomposition(Hamiltonian cycles) to obtain an eulerian circuit and chop the circuit into pieces(paths).

Remark 1.28.1. (Eulerian circuit) An eulerian circuit of a graph G is a circuit(vertices are possible to repeat) which uses up all the edges of G. It was proved by Euler in 1736 that an eulerian circuit of G exists if and only if G is a connected even graph i.e., every vertex of G is of even degree. This theorem clearly shows that finding an eulerian circuit of the "Seven-bridges graph" is impossible. Note that this theorem works for graphs with multi-edges and even loops.

The above theorem are not possible to be extended to star-decomposition. Clearly, we may decompose a complete graph into stars, but not any combination of sizes. For example, it is not possible to decompose K_7 into stars of sizes "6, 6, 4, 3, 2" even their sum is $21 = \|K_7\|$. (See it?)

But, for adding a constraint, the decomposition is possible.

Theorem 1.29. (Chiang Lin and Tay-Woei Shyu, JGT 1996) Let $m_1 \geq m_2 \geq \cdots \geq m_t$ such that $\sum_{i=1}^{t} m_i = \binom{n}{2}$. Then K_n can be decomposed into t stars of size $m_1, m_2, ..., m_t$ respectively if and only if $\sum_{i=1}^{k} m_i \leq \sum_{i=1}^{k}(n-i)$ for $k = 1, 2, ..., n-1$.

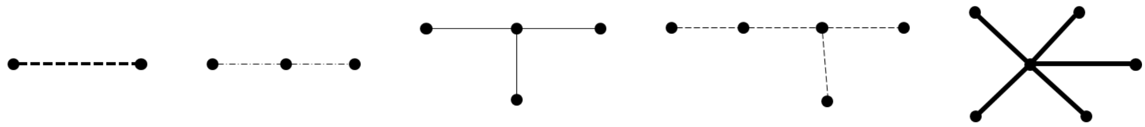
Example $< 6, 6, 4, 3, 2 >$ is not possible for K_7.

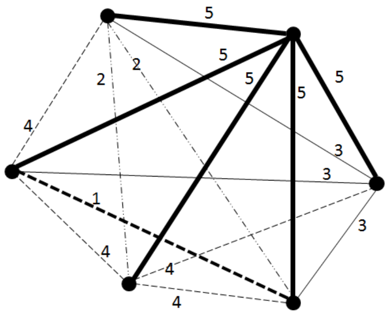
Choose $k = 2$, then $6 + 6 > (7-1) + (7-2)$.

The following problem is also known for sometime, but no one can solve it at this moment.

Problem 1.30. Given a collection of $n-1$ trees of size $1, 2, ..., n-1$ respectively, let them be $T_1, T_2, ..., T_{n-1}$. Prove that $K_n = T_1 + T_2 + \cdots + T_{n-1}$.

Example





Clearly, if all trees are either paths or stars, then we have the answers. But, how about the combination of these two types of graphs.

接下來，我們把焦點放在如何把完全圖或完全圖扣除一個1-factor分解成2-factors。(前者是基數點的完全圖，後者則為偶數點的完全圖。)相關的定理最著名的是Petersen's Theorem: 任意的正則偶圖都可以分解成2-factors(2-因子)。我們不難看出，上述定理所分割出來的2-因子並非全部都彼此同構；如果要加上這個要求，難度自然增高許多，尤其是這個2-因子是預先指定的圖(Prescribed 2-factor)。

Oberwolfach's Problem For each prescribed 2-factor R of K_n, there exist $\lfloor \frac{n-1}{2} \rfloor$ 2-factors which are isomorphic to R.
It has been proved in Theorem 1.7 that if R is a Hamilton cycle, then we have the answer of the above problem when n is odd. If n is even, then we do need more effort. (You may prove it yourself.) There are also several known results on special 2-factors.
A C_k-factor in K_n is a 2-factor in which each component is C_k, a cycle of length k. Clearly, a C_n-factor of K_n is a Hamilton cycle in K_n. Moreover, $k|n$.

Theorem 1.31. K_n can be decomposed into $\frac{n-1}{2}$ C_3-factors if and only if $n \equiv 3 \pmod 6$.

Proof. (Omitted here) The above theorem is equivalent to the following

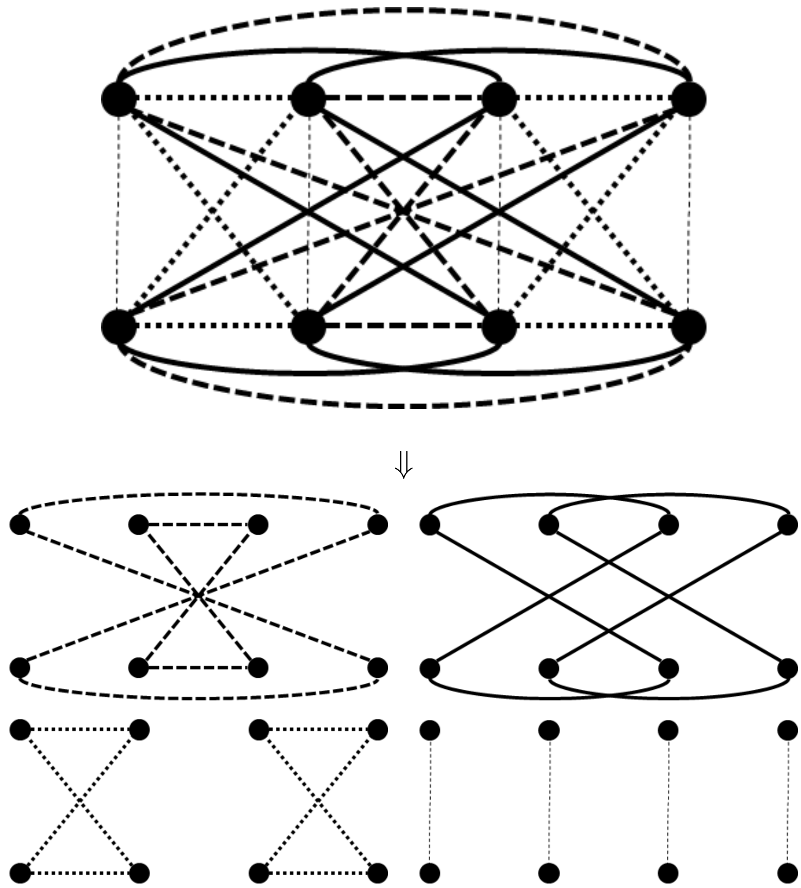
Theorem 1.32. A resoluble Steiner tuple system(Kirkman triple system) of order n exists if and only if $n \equiv 3 (\text{mod } 6)$.

This theorem is motivated by the well-known "Kirkman's 15 school girls problem" which asked to decompose K_{15} into 7 C_3-factors. In what follows, we claim that C_4-factors can be obtained in K_{4t}.

Theorem 1.33. K_n can be decomposed into a 1-factor and $\lfloor \frac{n-1}{2} \rfloor$ C_4-factors when ever $n \equiv 0 (\text{mod } 4)$.

Proof. Observe that $E(K_{4t}) = E(K_{2t}) \cup E(K_{2t,2t}) \cup E(K_{2t})$. By Theorem 1.3., K_{2t} can be decomposed into $2t - 1$ 1-factors namely $\{F_1, F_2, ..., F_{2t-1}\}$. Since there are two K_{2t}'s in corresponding order, we let $V_1(K_2t) = \{a_1, a_2, ..., a_{2t}\}$ and $V_2(K_{2t} = \{b_1, b_2, ..., b_{2t}\})$ and thus their corresponding 1-factorizations are $\{F_1, F_2, ..., F_{2t-1}\}$ and $\{F'_1, F'_2, ..., F'_{2t-1}\}$ respectively. Moreover, if $\{a_i, a_j\} \in F_k$, we let $\{b_i, b_j\} \in F'_k$. Now, we are ready to construct all the C_4-factors, there are $2t - 1$ C_4-factors. For each $k \in \{1, 2, ..., 2t - 1\}$, let $\langle F_k, F_{k'} \rangle$ be the set of t C_4's where $\langle F_k, F_{k'} \rangle = \{(a_i, b_j, b_i, a_j) | \{a_i, a_j\} \in F_k\}$. Now, it is a routine matter to check that $K_{4t} - I$ contains $2t - 1$ C_4-factors and indeed $2t - 1 = \lfloor \frac{4t-1}{2} \rfloor$. Note that $I = \{\{a_i, b_i\} | i = 1, 2, ..., 2t\}$.

Example



$K_8 \to I + C_4$-factors

Similar idea can be applied to find the factorization of C_6-factors. Note that we can use Theorem 1.32 to obtain this result. It is interesting to remark here that finding 3 C_6-factors in $K_{6,6}$ is not an easy task.

An extension of the study of Oberwolfach's problem is the so-called Hamilton-Waterloo problem.

Hamlton-Waterloo problem Find a 2-factorization of K_n(n is odd) in which r of its 2-factors are isomorphic to a 2-factor S of K_n with $r + s = \frac{n-1}{2}$. If n is even, then we consider the 2-factorization of $K_n - I$ where I is a 1-factor of K_n.

So far, the most popular study on this problem is to consider the case when R is Hamilton cycle and S is a cycle factor especially when S is a C_3-factor.

The following problem remains unsolved in general.

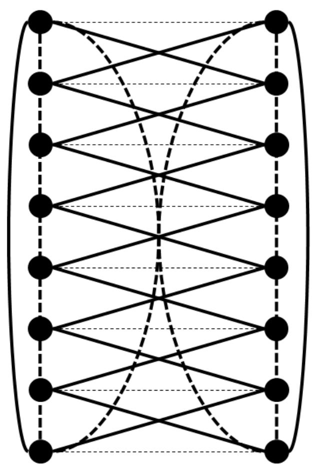
Open problem Find a 2-factorization of $K_n - \tilde{C}$ with all 2-factors isomorphic to a

C_3-factor where $\tilde{C} = C_n$ if n is odd and $\tilde{C} = C_n \cup I$ if n is even.

For convenience, we use $HW(n; r, s, ; m, k)$ to denote the case R is a C_n-factor and S is a C_k-factor. Therefore, the problem mentioned above is a special case of H-W problem: $HW(n; 1, \frac{n-3}{2}; n, 3)$. If we let $k = 4t$ where t is a positive integer, then $HW(n; r, s; n, 4k)$ has been solved.

Theorem 1.34. For positive integer k, an $HW(n; r, s; n, 4k)$ exists if and only if $r + s = \lfloor \frac{n-1}{2} \rfloor$ and $n \equiv 0 (\mathrm{mod}\ 4k)$ if $s > 0$ or $n \geq 3$ if $s = 0$.

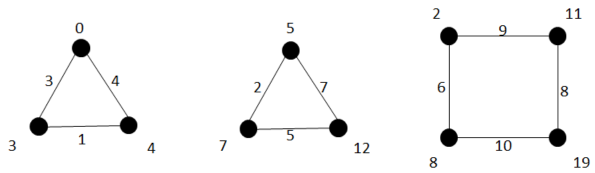
Proof. The necessity part is easy to see since "$s > 0$" implies that a C_{4k}-factor exists and thus $n \equiv (\mathrm{mod}\ 4k)$. On the other hand if $s = 0$, then K_n or $K_n - I(n$ even$)$ can be decomposed into Hamilton cycles. To prove the sufficiency, it takes more effort. (See reference.) In what follows, we use an example to explain the main idea. Let $n = 16$. It is not difficult to see that $K_n - I$ can be decomposed into 7 C_4-factors, see the following figure. Each matching to the left handside corresponds to a C_4-factor of K_{16}, and there are 7 1-factors of K_8. Now, we shall combine C_4-factors to obtain different number of Hamilton cycles from 0 to 7. Here is an example of "2". For others, you may be able to figure them out by similar idea.(?)



I is the set of edges with "dots".

A weaker version of Oberwolfach's problem is to decompose K_{2n+1} into isomorphic 2-regular subgraph of order n is therefore of size n. For example, we may decompose K_{21} into 2-regular graph $C_3 \cup C_3 \cup C_4$ and there are 21 members. The idea of this type of decomposition can be obtained by finding a ρ-labeling which uses 10 distinct half-differences.

An example is as following: $(V(K_{21}) = Z_{21})$



This problem is not solved yet. But, we can do something on this problem.

Theorem 1.35. Let $m_1, m_2, ..., m_k$ be positive integers not less then 3 and $n = \sum_{i=1}^{k} m_i$. Then K_{2n+1} can be decomposed into $2n+1$(not necessary be 2-regular graphs) $\bigcup_{i=1}^{k} C_{m_i}$.

Proof. (See DM 282(2004), 267-273.) Note that each cycle of length m_i does occur exactly $2n+1$ times, but $C_{m_1}, C_{m_2}, ..., C_{m_k}$ may not be vertex-disjoint(in this paper).

Chapter 2 Ascending Subgraph Decomposition

The idea of ascending subgraph decomposition was introduced around 25 years ago by several graph theories including P. Erdös. In fact, at 1987, there are two groups of researchers presented their works in the proceedings of Southeastern Conference on Combinatorics. Following their pioneered results, a few follow ups have been obtained. But, so far, it seems that proving the original conjecture is still very far from solved. This conjecture is considered as a "true" conjecture, i.e., it remains a good proof instead of considering to give a counter-example. This can be seen from the follows.

In this chapter, we shall use $G_1, G_2, ..., G_k$ to denote a sequence of subgraphs of G such that $G = G_1 + G_2 + \cdots + G_k$ and $G_i \leq G_{i+1}$ for $i = 1, 2, ..., k-1$. Now, we start with the introduction of the original version of conjecture proposed in the beginning.

Definition 2.1. If $G = G_1 + G_2 + \cdots + G_k$ and $G_i \leq G_{i+1}$ ($\|G_i\| < \|G_{i+1}\|$), then G is said to have an ascending subgraph decomposition(ASD in short).

Conjecture 2.2. If G is a graph such that $\binom{n+1}{2} \leq \|G\| \leq \binom{n+2}{2}$, then G has a ASD: $G = G_1 + G_2 + \cdots + G_n$.

It is not difficult to see that several special classes of graphs do have an ASD, for example complete graphs, Petersen graph, stars, paths, cycles(or 2-regular graph), ..., etc. The first non-trivial class of graphs which have an ASD mentioned in 1987 is a star forest. (A star forest is forest with its components stars.)

One of the well-known techniques in proving Conjecture 2.2. is the following. If $\|G\| = \binom{n+1}{2}$, then we do nothing but proving G has an ASD such that $\|G_i\| = i$ for $i = 1, 2, ..., n$. On the other hand, if $\|G\| > \binom{n+1}{2}$, then let $\tilde{G}$ be a graph obtained by deleting $\|G\| - \binom{n+1}{2}$ edges form G, i.e., $\tilde{G}$ is a subgraph of G such that $\|\tilde{G}\| = \binom{n+1}{2}$. Now, again, we prove $\tilde{G}$ has an ASD $\tilde{G} = \tilde{G_1} + \tilde{G_2} + \cdots + \tilde{G_n}$ with $\|G_i\| = i$. By adding all the deleted edges to $\tilde{G_n}$, we prove that G has an ASD and this verify Conjecture 2.2.

So, without mention otherwise, we consider the ASD of a graph of size $\binom{n+1}{2}$.

Theorem 2.3. Let G be a graph with $\Delta(G) \leq \frac{n+1}{2}$. Then G has an ASD.

Proof. First, we consider the case n is odd. Then, G has an edge-coloring which uses $\frac{n+1}{2}$ colors. By Theorem 1.6.9., we can color the graph G with $\frac{n+1}{2}$ colors such that each color class has exactly n edges. This implies that G can be decomposed into $\frac{n+1}{2}$ matchings each has n edges. Hence, G has an ASD which is easy to see. (?) On the other hand, let n be even. Then, $\Delta(G) \leq \frac{n-2}{2}$ and thus G has a edge coloring(equitable) which uses $\frac{n}{2}$ colors such that each color class has $n + 1$ edges. Hence, the proof follows by assigning matchings for $G_1, G_2, ..., G_n$. ■

In fact, we can allow $\Delta(G) \leq \frac{n+3}{2}$ and obtain a similar conclusion. (See it?) This idea of proof is related to an integer-partition problem.

Definition 2.4. (n-admissible) A set of t positive integers $m_1 \geq m_2 \geq \cdots \geq m_t$ is said to be n-admissible if $\sum_{i=1}^{t} m_i = \binom{n+1}{2}$.

We shall denote an n-admissible t-tuple by $\langle m_1, m_2, ..., m_t \rangle$.

Definition 2.5. (n-realizable) An n-admissible t-tuple $\langle m_1, m_2, ..., m_t \rangle$ is said to be n-realizable if there exits a partition of the set $[1, n] = \{1, 2, ..., n\}$ into t subsets $S_1, S_2, ..., S_t$ such that $\sum_{x \in S_i} x = m_i$ for $i = 1, 2, ..., t$.

A natural question about this study is to determine which n-admissible t-tuple is n-realizable. For example, from the proof of Theorem 2.3, we notice that $\langle n, n, ..., n \rangle$ ($\frac{n+1}{2}$-tuple) and $\langle n+1, n+1, ..., n+1 \rangle$ ($\frac{n}{2}$-tuple) are n-realizable. But, it is easy to see that $\langle 6, 5, 5, 2, 2, 2 \rangle$ is 6-admissible but not 6-realizable.

Motivated by decomposing a star forest of size $\binom{n+1}{2}$ into ascending "star" decomposition, P. Erdös conjectured that an n-admissible t-tuple $\langle m_1, m_2, ..., m_t \rangle$ is n-realizable provided that $m_t \geq n$. This conjecture was proved to be ture by Ma et al. and the paper appeared in Combinatorica(1994). Later, this result was further improved as follows:

Theorem 2.6. (陳福龍等四人) A n-admissible t-tuple $\langle m_1, m_2, ..., m_t \rangle$ is n-realizable provided $m_{t-1} \geq n$.

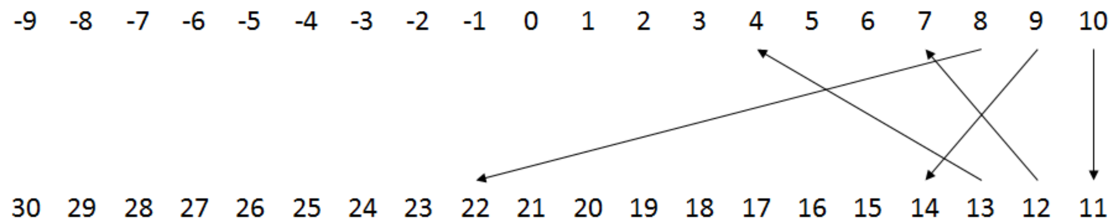
We use an example to explain the whole idea.

Example $\langle 30, 23, 21, 19, 17, 10 \rangle$ is 15-realizable.

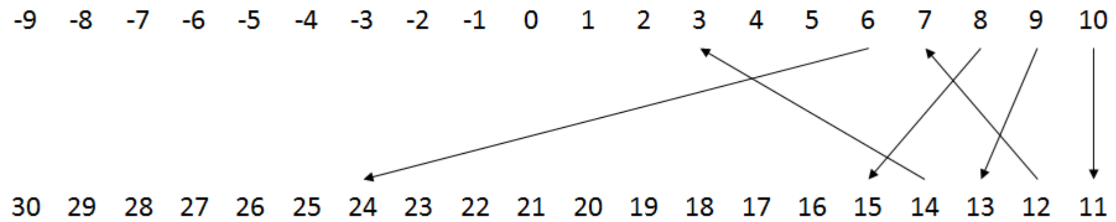
Fact 1. We may assume $m_{t-1} \geq n+1$.

Fact 2. For may assume $\langle m_1, m_2, ..., m_t \rangle$ we may let $l = n - 2(t-1)$, $N = 2t + 2l - 1$ and $M = m_1$.

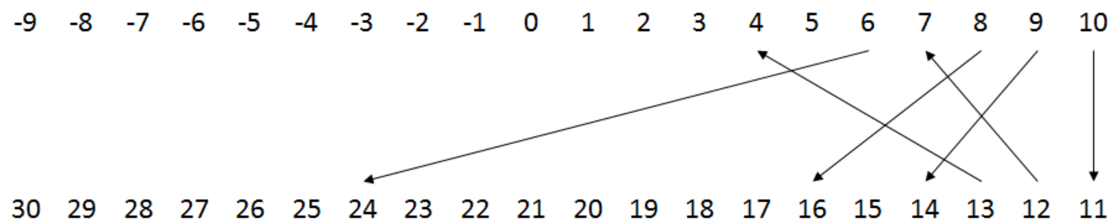
Since $n > 2(k-1)$, $l \geq 1$ and $N = n + l + 1 \geq n + 2$. We start with the following figure. Let $A = \{30, 23, 21, 19, 17\}$,



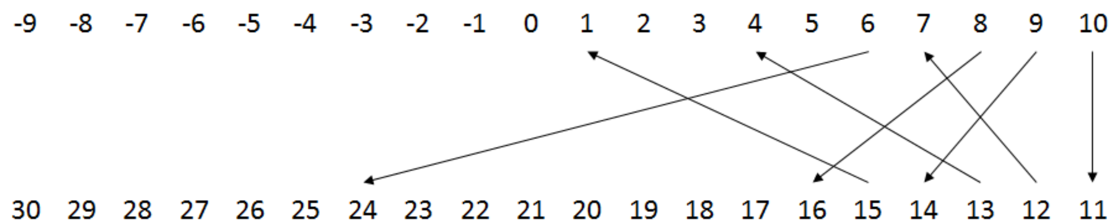
Add $\{22\}$, $A_1 = \{30, 23, 22, 21, 19, 17\}$,



Delete $\{22\}$, add $\{24\}$, $A_2 = \{30, 24, 23, 21, 19, 17\}$,



Add $\{16\}$, $A_3 = \{30, 24, 23, 21, 19, 17, 16\}$,



$A_4 = \{30, 24, 23, 21, 19, 17, 16\}$, $A_4 = A_3$, stop!

$$30 = 6 + 24 = 6 + 8 + 16 = 6 + 8 + 1 + 15$$

顯然要刻劃那一類n-admissible的$\langle m_1, m_2, ..., m_t\rangle$是$n$-realizable並不容易；但是，知道有那些$t$-tuple是可以實現的序列，對於ASD的研究會有很大的助益。

Theorem 2.7. If G is of size $\binom{n+1}{2}$ and G can be decomposed into paths of lengths $m_1, m_2, ..., m_t$ such that $\langle m_1, m_2, ..., m_t\rangle$ is n-realizable, then G has an ASD with each member a path.

Proof. It follows by cutting the t paths properly. ■

For example, the Petersen graph can be decomposed into paths of length 5, 4, 3, 2, 1 and thus we have an ASD with each member a path. A graph with the above property is called "path-perfect".

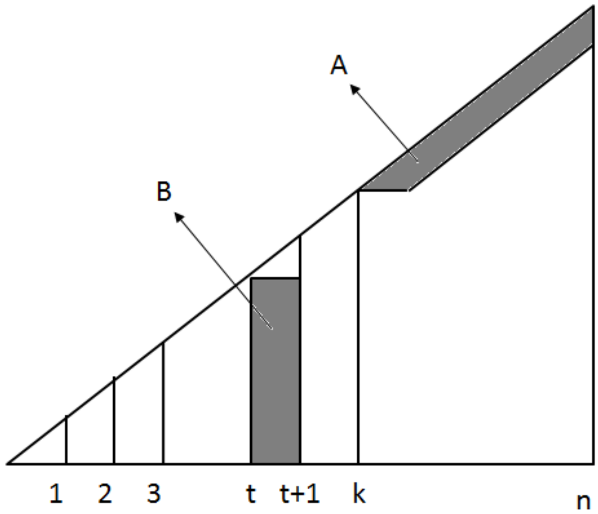
Problem. Determine all complete multipartite graphs which are path-perfect.

Definition 2.8. A graph G is called an $\langle m_1, m_2, ..., m_t\rangle$ star (matching) decomposable if there exist t edge-disjoint stars (matchings) $Z_1, Z_2, ..., Z_t$ such that they form a partition of $E(G)$ and $\|Z_i\| = m_i$, $i = 1, 2, ..., t$.

Now, it is not difficult to see the following.

Theorem 2.9. If G has an $\langle m_1, m_2, ..., m_t\rangle$ star (matching) decomposition such that $\langle m_1, m_2, ..., m_t\rangle$ is n-realizable, then G has an ASD with each member a star(matching).

Proof. Select stars from $Z_i, i = 1, 2, ..., t$. As a consequence, in order to have a star ASD, there must exist at least one vertex whose degree is at least n. This show that Theorem 2.8 is more friendly. But, a similar reason show that G must have paths which are of length at least n. Based on the above observation, another technique must be developed in order that we can handle all different classes of graphs. We start with a figure. The triangle represents an ASD of G of size $\binom{n+1}{2}$, and the vertical lengths are the sizes of members $G_1, G_2, ..., G_n$. If we cut off a piece of the triangle, say A, then the decomposition



becomes to be subgraphs of sizes $1, 2, ..., k, k, k+1, ..., n-1$. On the other hand, if we take B away, then the decomposition turns out to be of type $1, 2, ..., t-1, t+1, t+2, ..., n$. These two ideas are obtained by W.H.Hu in early 90s. We introduce the above ideas in what follows.

Definition 2.10. Let G be graph with $\binom{n+1}{2} - l$ edges, $0 \leq l \leq n-1$. If the edge set of G can be partitioned into n sets generating graphs $G_1, G_2, ..., G_n$ such that $G_i \leq G_{i+1}$ for $i = 1, 2, ..., n-1$, and

$$\|G_i\| = \begin{cases} i, & i = 1, 2, ..., n-l; \text{ and} \\ i-1, & i = n-l+1, ..., n, \end{cases}$$

then G has a nearly ascending subgraph decomposition (NASD).

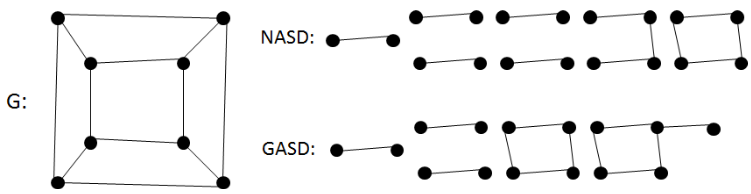
Definition 2.11. Let G be a graph with $\binom{n+1}{2} - t$ edges, $0 \leq t \leq n-1$. If $E(G)$ can be partitioned into n sets generating graphs $G_1, G_2, ..., G_{n-1}$ such that $G_i \leq G_{i+1}$ for

$i = 1, 2, ..., n-2$, and

$$\|G_i\| = \begin{cases} i, & i = 1, 2, ..., t-l; \text{ and} \\ i+1, & i = t, ..., n-1, \end{cases}$$

then G has a generalized ASD (GASD).

Example.



Remark. Both NASD and GASD extend the notion of ASD by letting $l, t = 1$ respectively. (See it?)

Theorem 2.12. Any forest G with $\binom{n+1}{2} - l$ edges, where $0 \le l \le n-1$, has an NASD.

證明內容可參考胡維新(1991)在交大應數系的博士論文，其中用到一個基本概念如下：

(Fact) If G is a graph with $\binom{n+1}{2} - l$ edges, where $0 \le l \le n-2$ and there exist n edges $e_1, e_2, ..., e_n$ such that $G \setminus \{e_1, e_2, ..., e_n\}$ has an NASD with members , moreover is vertex-disjoint with , then G had an NASD.

(Fact) If G a graph with $\binom{n+1}{2} - (n-1)$ edges, and there exist n edges $e_1, e_2, ..., e_n$ such that $G \setminus \{e_1, e_2, ..., e_n\}$ had an ASD with members $G'_1, G'_2, ..., G'_n$ moreover e_{i+1} is vertex-disjoint with G'_i, $i = 1, 2, ..., n-2$, then G has an NASD. ($\ll 1, 1, 2, ..., n-1 \gg$)

Corollary 2.13. Any forest has an ASD.

The following two results are also known.

Theorem 2.14. (DM 226, 397-402) Every complete multipartite graph has an ASD.

Theorem 2.15. (DM 253, 11-18) Every regular graph has an ASD.

Note that the above two classes of graphs are of size $\binom{n+1}{2} + t$ where $0 \leq t < n + 1$.

除了上述兩個已經發表的論文結果，後續的研究多半都是更特殊的圖論，其中有一些成果是證明當G非常接近完全圖時，該圖G有ASD.

Problem. Let H be a subgraph of K_m. Find H such that $K_m - H$ has an ASD.

Clearly, if H is of size small, then it is easier to prove $K_m - H$ has an ASD, but how small $\|H\|$ can be. Of course, if the conjecture is true, then H can be any subgraph of K_m. Therefore, this approach does not make sense to prove the conjecture, but at least we are able to verify the conjecture for more classes of graphs. The following class of graphs is of high importance in verifying the ASD conjecture in general.

Conjecture 2.16. Every bipartite graph has an ASD. Before we go to another chapter, we present a proof of showing an n-admissible sequence is n-realizable.

Proposition 2.17. Let m, l, k be positive integers such that $k \geq 0$, $m \geq 0$, $0 \leq l \leq \binom{n+1}{2}$ and $(k-1)m + l + \binom{k-1}{2} = \binom{n+1}{2}$. Then $\langle m, m+1, ..., m+k-2, l\rangle$ is n-realizable.

Proof. By induction on n.

First, if $m + k - 2 \leq n$, then the proof follows easily. On the other hand, consider $m + k - 2 > n$.

(1) $m \le n$

There exist a j, $1 \le j < k-1$ such that $m-j+1 = n$. Now, let $A_i = \{m+i-1\}$ for $i = 1, 2, ..., j$. Let $n' = m-1$. Then, it is clear that $\langle m+j, m+j+1, ..., m+k-2, l\rangle$ is n'-admissible. By induction, this sequence is n'-realizable. This concludes the proof.

(2) $m > n$

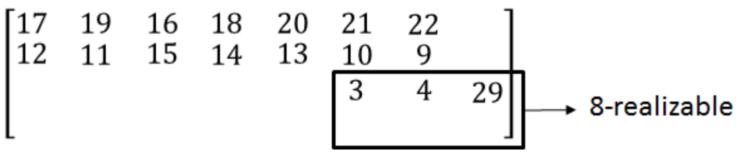
(2.1) $m \ge 2n - 2k + 3$

This implies that $m \ge n + (n - 2k + 3)$. Now, for $i = 1, 2, ..., k-1$, let $B_i = \{n+(i-1), n-2k+i+2\}$. The proof follows by induction, $\langle m-(2n-2k+3), m+1-(2n-2k+3), ..., m+k-2-(2n-2k+3), l\rangle$ is $(n-2k+2)$-realizable.

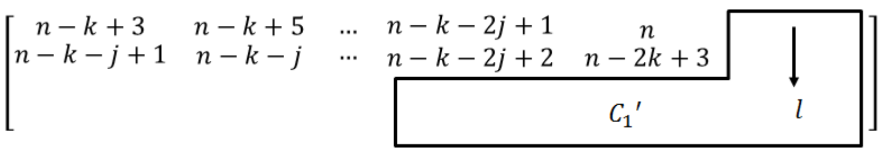
(2.2) $m < 2n - 2k + 3$ and $m + \lceil \frac{k-1}{2} \rceil - 1 \ge 2n - 2k + 3$

In this case, we can find j such that $m + j - 1 = 2n - 2k + 3$ and $1 \le j \le \lceil \frac{k-1}{2} \rceil$. Before we present the detail proof, let's look at an example where

$m = 29, k = 8, j == 3, l = 29$ and $n = 22$, i.e., $\langle 29, 30, 31, 32, 33, 34, 35, 29\rangle$.



Now, the partition array is as following



By induction $\langle m - 2k + 2k + 2j - 4, ..., m - 2n + 3k - 5, l\rangle$ is $(n - 2k + 2)$-realizable and the proof follows.

(3.1) $m + \lceil \frac{k-1}{2} \rceil - 1 < 2n - 2k + 3$ and $k - 1$ is odd

The solution follows by arraying the elements.

$$\left[\begin{array}{cccc|c} n-k+3 & n-k+5 & \cdots & n & L \\ m-n+k-3 & m-n+k-4 & \cdots & m-n+k-2 & \end{array}\right]$$

$$m \qquad m+1 \qquad m+k-2 \qquad l$$

Here L is the set of elements not occurred to its left, i.e., A; is the set of elements in ith column, $i = 1, 2, ..., k-1$.

(3.2) $m + \lceil \frac{k-1}{2} \rceil - 1 < 2n - 2k + 3$ and $k - 1$ is even

$$\left[\begin{array}{ccccc|c} n-k+3 & n-k+5 & \cdots & n & \cdots & n-1 \\ m-n+k-3 & m-n+k-4\cdots & & m-n+\frac{k-1}{2}-1\cdots & & m-n+k-1 \end{array} \;\; L \right]$$

This concludes the proof.