Week 5 April, 1st (Chapter 6)

Definition

Let A be a square matrix, real or complex. If $A\vec{x} = \lambda \vec{x}$ is true for some nonzero vector $\vec{x}$, then λ is an eigenvalue of A. The vector $\vec{x}$ is an eigenvector associated with the eigenvalue λ. ($\vec{x}$ may be a complex vector, i.e, with complex components and $\lambda \in \mathbb{C}$.)

e.g. $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{x}} = \underset{\lambda}{3} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underset{\lambda}{1} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

A has two eigenvalues 1 and 3.

Eigenvalue, characteristic values, latent roots, eigenwerte.

Fact 1 An eigenvector of a square matrix A associated with exactly one eigenvalue. (This is not true on the converse implication!)

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 3 \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

(*) In fact, if $A\vec{x} = \lambda\vec{x}$, then $A(\mu\vec{x}) = \lambda(\mu\vec{x})$.

Proof. (Fact 1)

Let $A\vec{v} = \lambda_1\vec{v}$ and $A\vec{v} = \lambda_2\vec{v}$ where $\vec{v}$ is an eigenvector of A. $A\vec{v} - A\vec{v} = (\lambda_1 - \lambda_2)\vec{v} = \vec{0} \Rightarrow \lambda_1 = \lambda_2$.

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

eigenvalue

So, how to find eigenvalues of A?

Two points of View :

1. $A\vec{x} - \lambda\vec{x} = \vec{0}$

$\Rightarrow (A - \lambda I)\vec{x} = \vec{0}$

A system of equations which has a nontrivial solution.

$\Rightarrow \det(A - \lambda I) = 0$

characteristic equation of A

Also, denoted by $P(\lambda; A)$ sometimes!

2. Consider $A - \lambda I$ as a linear transformation from $\mathbb{R}^n$ (or $\mathbb{C}^n$) to $\mathbb{R}^n$ (or $\mathbb{C}^n$).

Then, $\vec{x}$ is in $\mathrm{Null}(A-\lambda I)$, as a matter of fact $\mathrm{Null}(A-\lambda I) \neq \{\vec{0}\}$ if λ is indeed an eigenvalue of A. (?)

$$A = \begin{bmatrix} 2 & -5 & 5 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

eigenvectors are $\{ (\alpha, \beta, \beta) \mid \alpha^2 + \beta^2 \neq 0 \}$

why?

$(1,0,0)$, $(0,1,1)$ are eigenvectors of A associated with $\lambda = 2$.

(If $\vec{v}_1$, $\vec{v}_2$ are eigenvectors of A associated with λ,

then $\alpha \vec{v}_1 + \beta \vec{v}_2$ is also an eigenvector of A ---- λ .) $(\alpha^2 + \beta^2 \neq 0)$

Proof.
$$A(\alpha \vec{v}_1 + \beta \vec{v}_2) = \alpha A \vec{v}_1 + \beta A \vec{v}_2$$
$$= \alpha \lambda \vec{v}_1 + \beta \lambda \vec{v}_2$$
$$= \lambda (\alpha \vec{v}_1 + \beta \vec{v}_2) .$$ ▨

Definition We use $E_A(\lambda)$ to denote the vector subspace associated with λ.

e.g. $E_A(2)$ is of dim. 2.
(above example)