Linear Algebra (II), Week 2; 3,9 and 3,11. ①

Reminder

(*) Three tests are scheduled on

March 30th, May 11th and June 22nd.

(*) Three assignments will be handed out

on "two weeks" before test-dates.

Don't look at what you don't have,

look at what you have and make the

best out of it!

Linear Transformations (of a vector space)

Review

Mapping or Function

Let f be a mapping from A into B.

① f is 1-1 if $\forall x \in A$, $\exists 1\ y$, s.t. $f(x) = y$.

<u>Check</u>: Let $f(x_1) = f(x_2)$ $\forall x_1, x_2 \in A$.

Then $x_1 = x_2$.

② f is onto if $\forall y \in B$, $\exists x \in A$, s.t. $f(x) = y$.

<u>Check</u>: Let $f(A) = \{ f(x) \mid x \in A \}$.

Then $B \subseteq f(A)$.

③ f is a bijection (1-1 and onto) if f is 1-1 and onto.

Note that $|A| = |B|$ if and only if there exists a bijection from A onto B.

We say A and B has the same <u>cardinality</u>.

Question

Prove that $|\mathbb{N}| = |\mathbb{Z}|$ where $\mathbb{N}$ is the set of natural numbers and $\mathbb{Z}$ is the set of integers.

Proof. Let $f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even; and} \\ -\frac{x-1}{2} & \text{if } x \text{ is odd.} \end{cases}$

x	1	2	3	4	5	6	7	8	9	10	$\cdots \rightarrow$
$f(x)$	0	1	−1	2	−2	3	−3	4	−4	5	$\cdots$

Question

Prove that $|\mathbb{N}|$ and $|\mathbb{R}|$ are different or $|\mathbb{R}| > |\mathbb{N}|$.

Note To prove $|A| \leq |B|$ it is sufficient to show that there exists a 1-1 mapping from A into B.

If f is onto (from A onto B), then $|B| \leq |A|$.

(*) If f is 1-1 from A into B but not onto, then $|A| < |B|$.

Image of A' under $f: A \to B$ is $f(A')$ where $A' \subseteq A$.

Inverse image of B' under $f: A \to B$ is denoted by

$f^{-1}(B') = \{x \in A \mid f(x) \in B'\}$ where $B' \subseteq B$.

If $B' = \{b\}$, then $f^{-1}(B') =_{def} f^{-1}(b) = \{x \mid f(x) = b\}$.

(Instead of $f^{-1}(\{b\})$.)

Let U and V be two vector space over the same field F.

Definition (Linear Transformation)

A linear transformation T from U into V is a mapping $T: U \to V$ which has the properties:

(1) $T(\vec{u}_1 + \vec{u}_2) = T(\vec{u}_1) + T(\vec{u}_2)$, $\forall \vec{u}_1, \vec{u}_2 \in U$; and

(2) $T(\lambda \vec{u}) = \lambda T(\vec{u})$, $\forall \vec{u} \in U$ and $\lambda \in F$.

Definition (Linear Operator)

A linear transformation from V into itself is called a linear operator on V.

Check

$$T: U \to V \quad \text{(over } F\text{)},$$

(*) $T(\vec{u}_1 + \lambda \vec{u}_2) = T(\vec{u}_1) + \lambda T(\vec{u}_2)$.

(Combine (1) and (2) in definition.)

Example 1

$$T: \mathbb{R}^2 \to \mathbb{R}^2, \quad T(x,y) = (4x+5y, 6x-y).$$

Let $\vec{u}_1 = (x_1, y_1)$, $\vec{u}_2 = (x_2, y_2)$.

Then $T(\vec{u}_1 + \lambda \vec{u}_2) = T\big((x_1, y_1) + \lambda (x_2, y_2)\big)$

$= T(x_1 + \lambda x_2, y_1 + \lambda y_2)$

$= \big(4(x_1 + \lambda x_2) + 5(y_1 + \lambda y_2), 6(x_1 + \lambda x_2) - (y_1 + \lambda y_2)\big)$

$= (4x_1 + 5y_1 + \lambda(4x_2 + 5y_2), 6x_1 - y_1 + \lambda(6x_2 - y_2))$

$= (4x_1 + 5y_1, 6x_1 - y_1) + \lambda(4x_2 + 5y_2, 6x_2 - y_2)$

$= T(x_1, y_1) + \lambda(T(x_2, y_2))$.

(*) There are mappings which are not l. transf.

Example 2

$A = [a_{ij}]_{m\times n}$, $a_{ij} \in \mathbb{R}$. (Real matrix)

$\mathbb{R}_{n\times 1}$ ($n\times 1$ matrices over $\mathbb{R}$)

$T(X) = AX$ ($X \in \mathbb{R}_{n\times 1}$, then $AX \in \mathbb{R}_{m\times 1}$)

T is a linear transf. from $\mathbb{R}_{n\times 1}$ into $\mathbb{R}_{m\times 1}$.
(Both of them are v. sp.)

$$
\begin{aligned}
T(X+\lambda Y) &= A(X+\lambda Y) \\
&= AX + A(\lambda Y) \\
&= AX + \lambda AY \\
&= T(X) + \lambda T(Y). \quad \blacksquare
\end{aligned}
$$

Note This type of l. transf. is called a matrix transformation.

Example 3

Let U and V be vector spaces over F. Then the set of all l. transf. of U into V, $L(U,V)$, is a vector space over F.

Proof:

$$f, g : A \to B.$$

Define $(f+g)(x) = f(x) + g(x), \ \forall x \in A.$

$(\lambda f)(x) = \lambda f(x), \ \forall x \in A$ and $\lambda \in F$.

We have to define vector-addition, and scalar multiplication first.

To check $L(U,V)$ is a v. sp. over F we need to prove (1) ~ (10) of the def. of a vector sp..

(1) If T_1 and $T_2 \in L(U,V)$, then $T_1 + T_2 \in L(U,V)$.

⋮

(6) If $T \in L(U,V)$ and $\lambda \in F$, then

$\lambda T \in L(U,V)$.

⋮

Let $S \subseteq V$ be a subset of vectors in V.

$$\mathrm{Span}(S) = \left\{ \sum_{i=1}^{k} \lambda_i \vec{v}_i \;\middle|\; \vec{v}_i \in S,\ i = 1, 2, \cdots, k, \ \lambda_i \in F, \right\}.$$

Theorem

Let V be a vector space over F and S be a subset of V. Then $\mathrm{Span}(S)$ is a subspace of V.

Proof. It suffices to claim that $\forall\ \vec{u}, \vec{v} \in \mathrm{Span}(S)$ and $\lambda \in F$, $\vec{u} + \lambda \vec{v} \in \mathrm{Span}(S)$.

Let $\vec{u} = \sum_{i=1}^{h} \mu_i \vec{u}_i$ and $\vec{v} = \sum_{j=1}^{k} \lambda_j \vec{v}_j$.

$$\text{Then } \vec{u} + \lambda \vec{v} = \sum_{i=1}^{h} \mu_i \vec{u}_i + \lambda \left(\sum_{j=1}^{k} \lambda_j \vec{v}_j \right)$$

$$= \sum_{i=1}^{h} \mu_i \vec{u}_i + \sum_{j=1}^{k} (\lambda \lambda_j) \vec{v}_j .$$

Since u_i's and v_j's are in S, $\vec{u} + \lambda \vec{v} \in \mathrm{Span}(S)$.

This concludes the proof. $\blacksquare$

How to find a basis of V ?

Consider the case when $\dim V = n < +\infty$.
(finite)

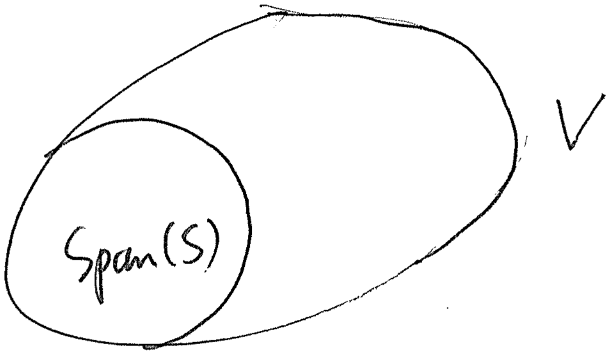
<u>Algorithm</u>

<u>Step 1.</u> Find a nonzero vector and put it in S.

<u>Step 2.</u> If there exist no nonzero vectors, then $\dim V = 0$ and we have no basis.

<u>Step 3.</u> Find a vector which is not in the $\mathrm{span}(S)$.

<u>Step 4.</u> Stop when $\mathrm{Span}(S) = V$ and then S is a basis.



<u>Theorem</u> If $|S| = \dim \mathrm{Span}(S)$, then for each vector $\vec{v} \in V$, $\vec{v} \notin \mathrm{Span}(S)$, $S \cup \{v\}$ is a linearly independent set.

<u>Proof.</u> Let $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ and $\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \cdots + \lambda_k \vec{v}_k + \lambda \vec{v} = \vec{0}$.

(1) $\lambda \neq 0 \Rightarrow \vec{v} \in \mathrm{Span}(S)$. $\rightarrow\leftarrow$

(2) $\lambda = 0 \Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$ and we have the proof. $\blacksquare$

Theorem

Let $\dim V = n$. Then any n vectors which span V must be a linearly independent set.

Proof. Let $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ spans V.

Suppose that S is linearly dependent.

Then, there exist scalars (not all zero) $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that $\sum_{i=1}^{n} \alpha_i \vec{v}_i = \vec{0}$.

Let $\alpha_j \neq 0$. Then, $\vec{v}_j = \frac{-1}{\alpha_j}\left(\sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i \vec{v}_i \right)$.

This implies that $V \subseteq \mathrm{Span}(S \setminus \{\vec{v}_j\})$. (?)

Since V can not be spanned by a set of $n-1$ or less vectors, a contradiction.

Why a vector space of dimension n can not be spanned by less than n vectors?

<u>Theorem</u> Any set S of n linearly independent vectors ~~set~~ in a vector space V of dimension n is a basis of V.

<u>Proof.</u> It suffices to prove that the set of those n vectors spans V.

Let $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ and S is not a basis of V.
Assume that $\mathrm{Span}(S) \subsetneq V$. Then, there exists a vector $\vec{v}$,
$\vec{v} = V \setminus \mathrm{Span}(S)$.
Then $S \cup \{\vec{v}\}$ is a linearly independent set which has $n+1$ vectors. But, this is not possible since $\dim V = n$.

(?)

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k l. independent ~~set~~ vectors span a vector subspace of V ~~of~~ which has dimension k.