Week 17 – Continued

Proof of Cramer's Rule

Let $A = [a_{ij}]$ and C^{ij} be its (i,j)–cofactor.
Then it is not difficult to check that

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C^{11} & C^{21} & & C^{n1} \\ C^{12} & C^{22} & \cdots & C^{n2} \\ \vdots & \vdots & & \vdots \\ C^{1n} & C^{2n} & & C^{nn} \end{bmatrix} = \det(A) \cdot I_n .$$

Therefore $A^{-1} = \dfrac{1}{\det(A)} \cdot \begin{bmatrix} & & \\ & C^{ij} & \\ & & \end{bmatrix}^T$,

Now, consider $A\vec{x} = \vec{b}$ where $\det(A) \neq 0$.
Then $A^{-1}\vec{b} = \vec{x}$ and therefore,

$$x_j = \left(b_1 C^{1j} + b_2 C^{2j} + b_3 C^{3j} + \cdots + b_n C^{nj}\right) \cdot \frac{1}{\det(A)}$$

$$= A_j(\vec{b}) / \det(A).$$ (用 $\vec{b}$ 取代第 j 行!) $\blacksquare$

How to find the determinant of

$$A_n = \begin{bmatrix} x & y & & & & \\ y & x & y & & 0 & \\ & y & x & y & & \\ & & y & x & y & \\ & 0 & & & \ddots & y \\ & & & & y & x \end{bmatrix}_{n\times n}.$$

<u>Sol.</u> We find the solution by using the expansion formula via 1^{st} row.

Let $\det(A_n) = d_n$.

Then, $d_n = x\,d_{n-1} - y^2 d_{n-2}$.

e.g.

$$\det\left(\begin{bmatrix} 1 & 3 & & & & \\ 3 & 1 & 3 & & 0 & \\ & 3 & 1 & 3 & & \\ & & 3 & 1 & 3 & \\ & 0 & & 3 & 1 & 3 \\ & & & & 3 & 1 \end{bmatrix}\right) = \begin{vmatrix} 1 & 3 & 0 & 0 & 0 \\ 3 & 1 & 3 & 0 & 0 \\ 0 & 3 & 1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 3 & 1 \end{vmatrix} -$$

$$3 \cdot 3 \cdot \begin{vmatrix} 1 & 3 & 0 & 0 \\ 3 & 1 & 3 & 0 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 3 & 1 \end{vmatrix} = \cdots.$$