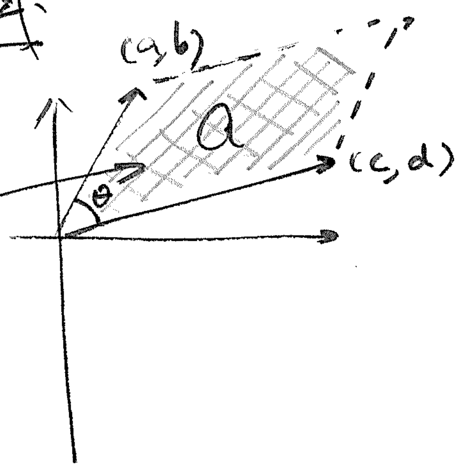
Week 17 1,5;1,7

Determinant 的几何意义

$$\left|\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right| = \text{area}$$



↖ 绝对值

Proof.

Let $\vec{v} = (a,b)$ and $\vec{u} = (c,d)$.

$$\mathcal{a} = |\vec{u}|\cdot|\vec{v}|\sin\theta$$

$$\mathcal{a}^2 = \left(|\vec{u}|\cdot|\vec{v}|\sin\theta\right)^2 = (a^2+b^2)(c^2+d^2)\sin^2\theta.$$

$$= (a^2+b^2)(c^2+d^2)(1-\cos^2\theta)$$

$$= (a^2+b^2)(c^2+d^2) - (a^2+b^2)(c^2+d^2)\cos^2\theta$$

$$= (a^2+b^2)(c^2+d^2) - \left(|\vec{u}||\vec{v}|\cos\theta\right)^2$$

$$= (a^2+b^2)(c^2+d^2) - \left(\vec{u}\cdot\vec{v}\right)^2$$

$$= (a^2+b^2)(c^2+d^2) - (ac+bd)^2$$

$$= a^2d^2 + b^2c^2 - 2abcd$$

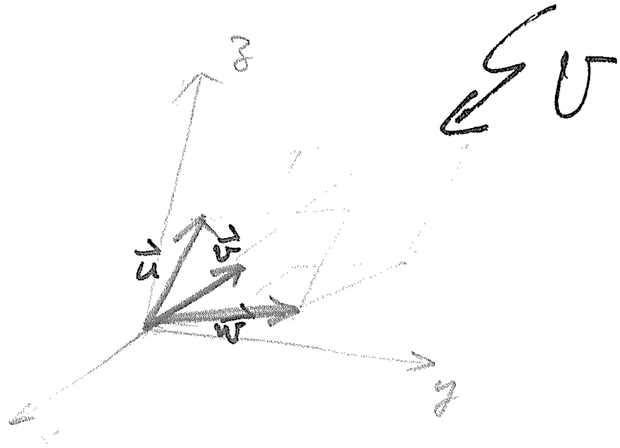
$$= (ad-bc)^2 = \left(\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right)^2$$

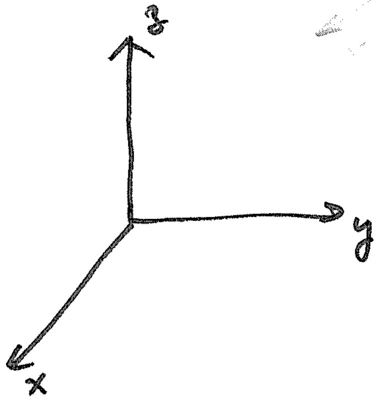
$$\left|\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right| = \mathcal{a}. \qquad \blacksquare$$

$\vec{u} = (a, b, c)$

$\vec{v} = (d, e, f)$

$\vec{w} = (g, h, i)$





3维空间 3個向量所圍成的平行六面体的体積。

(1) 由 $\vec{u}, \vec{v}$ 所圍成的面積(平行四边形).

(2) 与 $\vec{u}, \vec{v}$ 同時垂直的向量在 $\vec{w}$ 上的分量.

$$\vec{u} \times \vec{v} = \det\left(\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ d & e & f \end{bmatrix}\right) = (bf-ce, cd-af, ae-bd)$$

$$\Rightarrow \vec{w} \cdot (\vec{u} \times \vec{v}) = \det\left(\begin{bmatrix} g & h & i \\ a & b & c \\ d & e & f \end{bmatrix}\right).$$

体積 $V = \left|\det\left(\begin{bmatrix} g & h & i \\ a & b & c \\ d & e & f \end{bmatrix}\right)\right|.$ Proof 在下页

(*) $|\vec{a}\times\vec{b}|^2$

$= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$

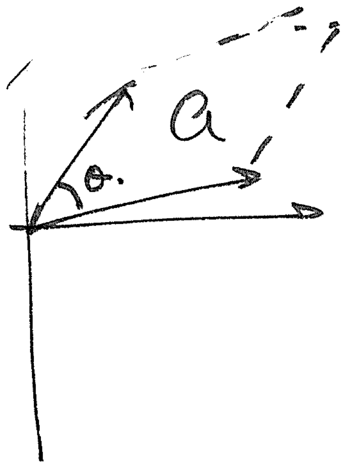
$= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1+a_2b_2+a_3b_3)^2$

$= |\vec{a}|^2|\vec{b}|^2 - \vec{a}\cdot\vec{b}^2$

$= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta$

$= |\vec{a}|^2|\vec{b}|\sin^2\theta.$

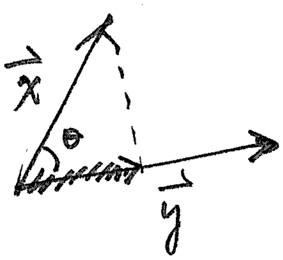
$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta.$ $\quad 180^\circ > \theta > 0^\circ$



$a = |\vec{a}\times\vec{b}|$

(2维，3维都可.)

(**)



$\vec{x}$在$\vec{y}$上的投影 $= \dfrac{\vec{y}}{|\vec{y}|}|\vec{x}|\cos\theta$

$= \dfrac{\vec{y}}{|\vec{y}|^2}|\vec{x}||\vec{y}|\cos\theta$

$= \dfrac{\vec{y}}{|\vec{y}|^2}(\vec{x}\cdot\vec{y}) = \dfrac{\vec{x}\cdot\vec{y}}{\vec{y}\cdot\vec{y}}\vec{y}$

Proof.

$\vec{u}, \vec{v}$ 所决定平行四边形面积为 $|\vec{u}\times\vec{v}|$。

$\vec{u}, \vec{v}, \vec{w}$ 所决定 V 为 $|\vec{u}\times\vec{v}|h$，其中 $\vec{h}$ 为 $\vec{w}$ 在 $\vec{u}\times\vec{v}$ 上的投影，所以

$$\vec{h} = \frac{\vec{w}\cdot(\vec{u}\times\vec{v})}{(\vec{u}\times\vec{v})\cdot(\vec{u}\times\vec{v})}\,\vec{u}\times\vec{v}.$$

$$\Rightarrow \quad V = \left|\frac{\vec{w}\cdot(\vec{u}\times\vec{v})}{(\vec{u}\times\vec{v})\cdot(\vec{u}\times\vec{v})}\,\vec{u}\times\vec{v}\right|\,|\vec{u}\times\vec{v}|$$

$$= |\vec{w}\cdot(\vec{u}\times\vec{v})| = \left|\det\left(\begin{bmatrix} g & h & i \\ a & b & c \\ d & e & f \end{bmatrix}\right)\right|. \quad \blacksquare$$

Week 17 (1,5;1,7)

Cofactor Expansion

$$
A = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}
\begin{array}{cc}
& \begin{array}{c} j\text{th column} \end{array} \\
& \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
 & & \cdots & \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix} & \begin{array}{c} \\ \\ i\text{th row} \\ \\ \end{array}
\end{array}
$$

M^{ij} ((i,j)-minor of A)

is obtained from A by removal of row i and column j from A.

(*) M^{ij} is a $(n-1)\times(n-1)$ matrix.

$$
A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad M^{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}.
$$

the (i,j)-cofactor of A is

Definition § Cofactor $C^{ij} = (-1)^{i+j} \det(M^{ij})$.

Definition

$$\det(A) = \sum_{j=1}^{n} a_{ij} C^{ij} \quad (\text{利用 } i\text{th row})$$

$$= \sum_{i=1}^{n} a_{ij} C^{ij} \quad (\text{利用 } j\text{th column}).$$

Example

$$A = \begin{bmatrix} -53 & 0 & 71 & 0 & -3 \\ 0 & 0 & 2 & 0 & 0 \\ 79 & 4 & -83 & 67 & -51 \\ -95 & 0 & 63 & 7 & -91 \\ 3 & 0 & -51 & 0 & 0 \end{bmatrix}$$

$$\det(A) = 2 \cdot (-1)^{2+3} \cdot \det\left(\begin{bmatrix} -53 & 0 & 0 & -3 \\ 79 & 4 & 67 & -51 \\ -95 & 0 & 7 & -91 \\ 3 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$= 2 \cdot (-1) \cdot 3 \cdot \boxed{(-1)} \boxed{3} \cdot \det\left(\begin{bmatrix} 0 & 0 & -3 \\ 4 & 67 & -51 \\ 0 & 7 & -91 \end{bmatrix} \right) = (-2)(-3)(-3) \cdot 28.$$

Cramer's Rule

Let $A_{n\times n}$ be an invertible matrix. Then the solution of $A\vec{x} = \vec{b}$ is given by

$$x_j = \frac{\det(A_j(\vec{b}))}{\det(A)}, \quad j = 1, 2, \cdots, n,$$

where $A_j(\vec{b})$ is the matrix obtained by replacing the entries in the jth column of A by $\vec{b}$.

Can you prove this rule?

Example

$$\begin{bmatrix} 1 & 4 & 2 \\ 3 & -3 & 6 \\ 2 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix}$$

$$x_1 = \det(A_1(\vec{b})) / \det(A)$$

$$= \det\left(\begin{bmatrix} 3 & 4 & 2 \\ 5 & -3 & 6 \\ -4 & 0 & 5 \end{bmatrix}\right) / -15 = -75/-15 = 5.$$

x_2, x_3.