Week 16 12,29; 12,31

Permutations (Continued)

Definition (Review)

Permutations

↙ ↘

odd permutations   even permutations

(Fact 1) There are $n!$ permutations from $[1,n]$ onto $[1,n]$.
(of $[1,n]$)

(Fact 2) There are $\frac{n!}{2}$ odd permutations and $\frac{n!}{2}$ even permutations.

It suffices to prove that $|O_n| = |E_n|$.

Proof. Let O_n (E_n) be the set odd (even) permutations of $[1,n]$

Define $\varphi : O_n \to E_n$ by $\varphi(\pi) = \pi \circ (12)$.

Then φ is 1-1 and onto.

(1-1) $\varphi(\pi_1) = \varphi(\pi_2) \Rightarrow \pi_1 \circ (12) = \pi_2 \circ (12)$

$\Rightarrow (\pi_1 \circ (12)) \circ (12) = (\pi_2 \circ (12)) \circ (12)$

$\Rightarrow \pi_1 = \pi_2$.

(onto) $\forall\ \pi \in E_n$, $\pi \circ (12) \in O_n$, and $\varphi(\pi \circ (12)) = \pi$. ∎

How to determine the parity of a permutation?

(Method 1)

A permutation can be written as the composition of disjoint cycles and a cycle of odd (even) length is an even (odd) permutation.

Therefore, if a permutation is written as the composition which has an odd number of even cycles, then it is an odd permutation.

e.g.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 3 & 5 & 4 & 7 & 6 \end{pmatrix} = \underline{(1\,2)}\circ(3)\circ\underline{(4\,5)}\circ\underline{(6\,7)} \quad \text{is odd.}$$

(Method 2)

A permutation $\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ j_1 & j_2 & j_3 & \cdots & j_n \end{pmatrix}$ is even (odd resp.)

if $\sum_{i=1}^{n}$ (number of larger integers precedes j_i) is even (odd resp.).

e.g.

even $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 3 & 4 & 5 & 2 \end{pmatrix}$: $0+1+1+1+1+4=8$,

odd $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 1 & 4 & 5 & 2 \end{pmatrix}$: $0+1+2+1+1+4=9$.

Theorem 1 $\det(A) = \det(A^T)$.

Proof. It suffices to prove that the inverse permutation of an even (odd) permutation is also an even (odd) permutation.

Since $\alpha \circ \alpha^{-1} = \cancel{(e)}\, e$ (identity) which is even, α and α^{-1} have the same parity. ! ∎

The following facts can be checked directly.

(1) Row operations E_1, E_2, E_3

- E_3: use $\alpha \vec{r}_i + \vec{r}_j$ for $\vec{r}_j$
- E_2: change two rows
- E_1: multiply a constant to a row

(2) $\det(A) = 0$ if

(i) one row is $\vec{0}$.

(ii) one column is $\vec{0}$,

(iii) one row is a multiple of the other, i.e., $\vec{r}_i = c\vec{r}_j$.

(iv) one column is a multiple of the other, i.e., $\vec{c}_i = \alpha \vec{c}_j$.

⋮