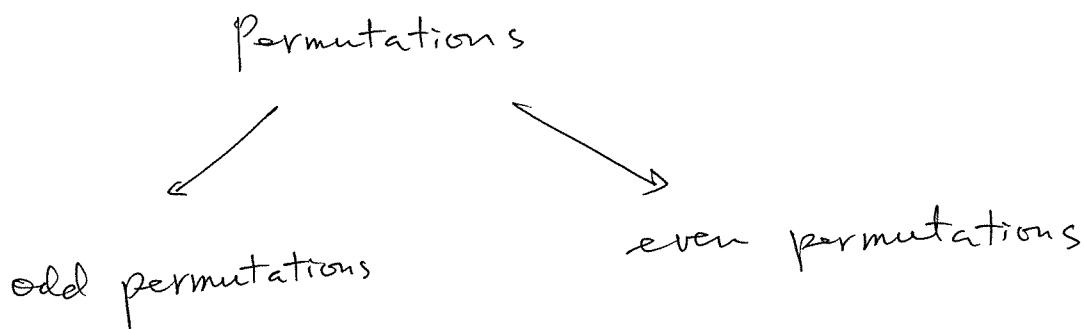


Permutations (Continued)

Definition (Review)



(Fact 1) There are $n!$ permutations from $[1, n]$ onto $[1, n]$
(of $[1, n]$)

(Fact 2) There are $\frac{n!}{2}$ odd permutations and $\frac{n!}{2}$ even permutations.

It suffices to prove that $|O_n| = |E_n|$.

Proof. Let O_n be the set of odd permutations of $[1, n]$
(E_n) (even)

Define $\varphi : O_n \rightarrow E_n$ by $\varphi(\pi) = \pi \circ (12)$.

Then φ is 1-1 and onto.

$$\begin{aligned} (1-1) \quad \varphi(\pi_1) = \varphi(\pi_2) &\Rightarrow \pi_1 \circ (12) = \pi_2 \circ (12) \\ &\Rightarrow (\pi_1 \circ (12)) \circ (12) = (\pi_2 \circ (12)) \circ (12) \\ &\Rightarrow \pi_1 = \pi_2. \end{aligned}$$

(onto) $\forall \pi \in E_n, \pi \circ (12) \in O_n$, and $\varphi(\pi \circ (12)) = \pi$. ▣

(2)

How to determine the parity of a permutation?

(Method 1)

A permutation can be written as the composition of disjoint cycles and a cycle of (odd length is an even permutation, (odd)

Therefore, if a permutation is written as the composition which has an odd number of even cycles, then it is an odd permutation.

e.g.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 3 & 5 & 4 & 7 & 6 \end{pmatrix}$$

$$= \underbrace{(12)} \circ \underbrace{(3)} \circ \underbrace{(45)} \circ \underbrace{(67)} \text{ is odd.}$$

(Method 2)

A permutation $\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ j_1 & j_2 & j_3 & \dots & j_n \end{pmatrix}$ is even (odd resp.)

if $\sum_{i=1}^n$ (number of larger integers precedes j_i) is even (odd resp.).

e.g.

$$\text{even } \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 3 & 4 & 5 & 2 \end{pmatrix} : 0+1+1+1+1+4=8,$$

$$\text{odd } \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 1 & 4 & 5 & 2 \end{pmatrix} : 0+1+2+1+1+4=9.$$

(3)

Theorem 1 $\det(A) = \det(A^T)$.

Proof: It suffices to prove that the inverse permutation of an even permutation is also an even permutation.
(odd) (odd)

Since $\alpha \circ \alpha^{-1} = \text{id}$ (identity) which is even, α and α^{-1} have the same parity. ! ■

The following facts can be checked directly.

(1) Row operations E_1, E_2, E_3 → use $\alpha \vec{r}_i + \vec{r}_j$ for \vec{r}_j
→ change two rows
→ multiply a constant to a row

(2) $\det(A) = 0$ if

(i) one row is $\vec{0}$.

(ii) one column is $\vec{0}$,

(iii) one row is a multiple of the other, i.e., $\vec{r}_i = c \vec{r}_j$.

(iv) one column is a multiple of the other, i.e.,
 $\vec{c}_i = \alpha \vec{c}_j$.

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